Math 105 Final Exam. (April 23, 2015)

1. (a) $1\cdot(x-1)-3(y-0)+2(z+1)=0$ or $x-3y+2z=-1$.

(b) circles.

(c) $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(y^3\cos 2x)\right)=\frac{\partial}{\partial y}\left(y^3(-2)\sin 2x\right)=3y^2(-2)\sin 2x$

$=-6y^2\sin 2x.$

(d) $a=2$, $b=6$, $n=4$.

(e) $\int_2^3 (6f-3g)\,dx = 6\int_2^3 f\,dx - 3\int_2^3 g(x)$

$=6(-1)-3\cdot 5=-21$

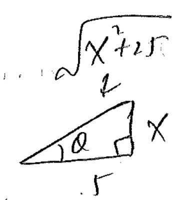
(f) $F(x)=\int f(x)\,dx=\int (x^3-\sin 2x)\,dx=\frac{x^4}{4}+\frac{1}{2}\cos 2x+c$

$1=F(0)=\frac{1}{2}\cos 0+c=\frac{1}{2}+c \Rightarrow c=\frac{1}{2}$. Hence,

$F(x)=\frac{x^4}{4}+\frac{1}{2}\cos 2x+\frac{1}{2}$

(g) $(\sin^6 x+8)\cos x$.

(h) $\int\frac{dx}{\sqrt{x^2+25}} \underset{x=5\tan\theta}{=} \int\frac{5\sec^2\theta\,d\theta}{\sqrt{(5\tan\theta)^2+25}}=\int\sec\theta\,d\theta$



$=\ln|\sec\theta+\tan\theta|+C=\ln\left|\frac{\sqrt{x^2+25}}{5}+\frac{x}{5}\right|+C.$

(i) $\int_0^{\pi/2} x\cos x\,dx \underset{\substack{u=x,\ v'=\cos x\\ u'=1,\ v=\sin x}}{=} (x\sin x)\Big|_0^{\pi/2}-\int_0^{\pi/2}\sin x\,dx$

$=\frac{\pi}{2}+\cos x\Big|_0^{\pi/2}=\frac{\pi}{2}+\left(\cos\frac{\pi}{2}-\cos 0\right)=\frac{\pi}{2}-1.$

(j) $\int\cos^3 x\,dx=\int\cos x\cos^2 x\,dx=\int\cos x(1-\sin^2 x)\,dx \underset{\substack{u=\sin x\\ du=\cos x\,dx}}{=} \int(1-u^2)\,du$

$=u-\frac{u^3}{3}+C=\sin x-\frac{1}{3}\sin^3 x+C.$

1. (k) $\int_0^1 \frac{x^4}{x^5-1}dx = \lim\limits_{t\to 1^-} \int_0^t \frac{x^4}{x^5-1}dx$

$= \lim\limits_{t\to 1^-} \left(\frac{1}{5}\ln|x^5-1|\right)\Big|_0^t = \lim\limits_{t\to 1^-} \frac{1}{5}\ln|t^5-1| = -\infty$,

so the integral is divergent

(l) $\left(\frac{1}{87}\right)/\left(1-\frac{1}{8}\right)$.

(m) $\frac{dy}{dt} = e^{y/3}\cos t \Rightarrow e^{-y/3}dy = \cos t\, dt$

$\int e^{-y/3}dy = \int \cos t\, dt \Rightarrow -3e^{-y/3} = \sin t + C$

$e^{-y/3} = \ln\left(-\frac{1}{3}\sin t - \frac{C}{3}\right) \Rightarrow y = -3\ln\left(-\frac{1}{3}\sin t - \frac{C}{3}\right)$.

(n) $\lim\left(\ln\left(\sin\frac{1}{n}\right) + \ln 2n\right) = \lim \ln\left(2n \sin\frac{1}{n}\right)$

$= \lim \ln\left(2 \cdot \frac{\sin\frac{1}{n}}{\frac{1}{n}}\right) = \ln 2$.

2. (a) $\displaystyle\int \frac{e^x}{(e^x+1)(e^x-3)}\,dx \underset{\substack{u=e^x \\ du=e^x dx}}{=\!=} \int \frac{du}{(u+1)(u-3)}.$

$$\frac{1}{(u+1)(u-3)} = \frac{A}{u+1} + \frac{B}{u-3} \Rightarrow 1 = A(u-3)+B(u+1)$$

$u=-1 \Rightarrow \quad 1 = A(-4) \Rightarrow A = -1/4.$

$u=3 \Rightarrow \quad 1 = B(4) \Rightarrow B = 1/4.$

Hence, $\displaystyle\int \frac{e^x}{(e^x+1)(e^x-3)}\,dx = \int \left(\frac{-1/4}{u+1} + \frac{1/4}{u-3}\right) du$

$= -\frac{1}{4}\ln|u+1| + \frac{1}{4}\ln|u-3| + C = -\frac{1}{4}\ln|e^x+1| + \frac{1}{4}\ln|e^x-3| + C.$

(b) $\displaystyle\int_2^4 \frac{x^2-4x+4}{\sqrt{12+4x-x^2}}\,dx = \int_2^4 \frac{x^2-4x+4}{\sqrt{12-(x^2-4x)}}\,dx = \int_2^4 \frac{x^2-4x+4}{\sqrt{12-(x^2-4x+4-4)}}\,dx$

$\displaystyle = \int_2^4 \frac{(x-2)^2}{\sqrt{16-(x-2)^2}}\,dx \underset{\substack{x-2=4\sin\theta \\ dx=4\cos\theta}}{=\!=} \int_0^{\pi/6} \frac{(4\sin\theta)^2}{\sqrt{16-(4\sin\theta)^2}} \cdot 4\cos\theta\,d\theta$

$\displaystyle = 4^2 \int_0^{\pi/6} \sin^2\theta\,d\theta = 16\int_0^{\pi/6} \frac{1-\cos 2\theta}{2}\,d\theta = 8\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/6} = 8\left(\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right).$

3. (a) Let $g(x,y)=x^2+y^2-4$. Then we need to solve the system

$$\begin{cases} f_x=\lambda g_x \\ f_y=\lambda g_y \\ g=0 \end{cases} \quad \text{or} \quad \begin{cases} 2(x-1)=\lambda(2x), & (1) \\ 2(y+1)=\lambda(2y) & (2) \\ x^2+y^2=4 & (3) \end{cases}$$

$(1)\,y \Rightarrow \quad xy-y=\lambda xy \quad (4)$

$(2)\,x \Rightarrow \quad xy+x=\lambda xy \quad (5)$

(4) and $(5) \Rightarrow \quad y=-x. \quad (6)$

By (3) and (6), we get $x^2+(-x)^2=4 \Rightarrow x=\pm\sqrt{2} \Rightarrow y=\mp\sqrt{2}$.

$f(\sqrt{2},-\sqrt{2})=(\sqrt{2}-1)^2+(-\sqrt{2}+1)^2=2(\sqrt{2}-1)^2=2(2-2\sqrt{2}+1)=6-4\sqrt{2}$.

$f(-\sqrt{2},\sqrt{2})=(-\sqrt{2}-1)^2+(\sqrt{2}+1)^2=2(\sqrt{2}+1)^2=2(2+2\sqrt{2}+1)=6+4\sqrt{2}$.

Hence, the maximum value of $f(x,y)$ on the circle is $6+4\sqrt{2}$, the minimum value of $f(x,y)$ on the circle is $6-4\sqrt{2}$.

(b) $\begin{cases} \dfrac{\partial f}{\partial x}=2(x-1)=0 \\ \dfrac{\partial f}{\partial y}=2(y+1)=0 \end{cases} \Rightarrow \quad (1,-1)$ is the only critical point. $\quad f(1,-1)=(1-1)^2+(-1+1)^2=0. \quad (7)$

By the conclusions in (a) and (7), we know that the maximum value of $f(x,y)$ on the region R is $6+4\sqrt{2}$, the minimum value of $f(x,y)$ on the region R is 0.

4. (a) $E(X)=\int_{-\infty}^{\infty} x f(x)\,dx = \int_{-1}^{1} x\left(\frac{1}{4}+\frac{1}{2}|x|\right)dx$

$= \int_{-1}^{0} x\left(\frac{1}{4}-\frac{1}{2}x\right)dx + \int_{0}^{1} x\left(\frac{1}{4}+\frac{1}{2}x\right)dx$

$= \int_{-1}^{0}\left(\frac{1}{4}x-\frac{1}{2}x^2\right)dx + \int_{0}^{1}\left(\frac{1}{4}x+\frac{1}{2}x^2\right)dx$

$= \left(\frac{1}{4}\frac{x^2}{2}-\frac{1}{2}\cdot\frac{1}{3}x^3\right)\Big|_{-1}^{0} + \left(\frac{1}{4}\frac{x^2}{2}+\frac{1}{2}\cdot\frac{1}{3}x^3\right)\Big|_{0}^{1}$

$= -\left(\frac{1}{8}+\frac{1}{6}\right)+\left(\frac{1}{8}+\frac{1}{6}\right)=0.$

(b) $F(x)=\int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{-1} f(t)\,dt + \int_{-1}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt$

$= 0 + \int_{-1}^{0}\left(\frac{1}{4}-\frac{1}{2}t\right)dt + \int_{0}^{x}\left(\frac{1}{4}+\frac{1}{2}t\right)dt$

$= \left(\frac{1}{4}t-\frac{1}{2}\frac{t^2}{2}\right)\Big|_{-1}^{0} + \left(\frac{1}{4}t+\frac{1}{2}\frac{t^2}{2}\right)\Big|_{0}^{x}$

$= -\left(-\frac{1}{4}-\frac{1}{4}\right)+\frac{1}{4}x+\frac{1}{4}x^2 = \frac{1}{2}+\frac{1}{4}x+\frac{1}{4}x^2.$

5. (a) $f'(x) = x \cdot \dfrac{1}{1-(-3x^3)} = x \displaystyle\sum_{n=0}^{\infty} (-3x^3)^n = \sum_{n=0}^{\infty} (-1)^n 3^n x^{3n+1}$

$f(x) = \displaystyle\int \left(\sum_{n=0}^{\infty} (-1)^n 3^n x^{3n+1} \right) dx = C + \sum_{n=0}^{\infty} (-1)^n 3^n \frac{x^{3n+2}}{3n+2}$.

With $f(0)=1$, we have $C=1$ so $f(x) = 1 + \displaystyle\sum_{n=0}^{\infty} (-1)^n 3^n \frac{x^{3n+2}}{3n+2}$.

(b) $\displaystyle\lim_{n\to\infty} \frac{(n^2+n+1)/(n^5-n)}{(1/n^3)} = \lim \frac{n^5+n^4+n^3}{n^5-n} =$

$= \lim \dfrac{1+(1/n)+(1/n^2)}{1-\left(\frac{1}{n^4}\right)} = 1.$ Since $\sum \frac{1}{n^3}$ converges,

$\displaystyle\sum_{n=0}^{\infty} \frac{n^2+n+1}{n^5-n}$ converges.

(c) Since $\dfrac{3m+\sin\sqrt{m}}{m^2} > \dfrac{3m-1}{m^2}$ and

$\sum \frac{3m-1}{m^2}$ diverges, $\sum \frac{3m+\sin\sqrt{m}}{m^2}$ diverges

(or $\dfrac{3m+\sin\sqrt{m}}{m^2} > \dfrac{3m-m}{m^2} = \dfrac{2m}{m^2} = \dfrac{2}{m}$)

(d) $f(x) = \dfrac{1}{x(\ln x)^3}$ is positive, decreasing and continuous for $x \geq 2$. Since

$\displaystyle\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^3}\,dx$

$\underset{\substack{u=\ln x \\ du=\frac{dx}{x}}}{=} \displaystyle\lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{du}{u^3} = \lim_{t\to\infty} \left(-\frac{1}{2u^2}\right)\Big|_{\ln 2}^{\ln t} = \frac{1}{2(\ln 2)^2}$,

the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$ converges.

6. (a) Since $\sum_{n=0}^{\infty}(1-a_n)$ converges, $\lim(1-a_n)=0$ or $\lim a_n = 1$. Hence, $\lim(2^n a_n)=\infty$, which implies that $\sum 2^n a_n$ diverges.

(b) For $|x|>1$, $\lim|a_n x^n| = \lim(a_n |x|^n) = \infty$

Hence, $\lim a_n x^n \neq 0$. This proves that

$\sum a_n x^n$ diverges for $|x|>1$. (1).

Also, for $|x|<1$ with $x\neq 0$, we have

$$\lim \left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim\left(\frac{a_{n+1}}{a_n}|x|\right) = \frac{\lim a_{n+1}}{\lim a_n}\cdot|x| = |x| < 1$$

$\Rightarrow$ $\sum a_n x^n$ converges for $|x|<1$ (2)

By (1) and (2), the radius of convergence is 1.