Full Solutions
MATH105 April 2014

April 22, 2015

How to use this resource

- When you feel reasonably confident, simulate a full exam and grade your solutions. This document provides full solutions that you can use to grade your work.
- If you're not quite ready to simulate a full exam, we suggest you thoroughly and slowly work through each problem. To check if your answer is correct, without spoiling the full solution, we provide a pdf with the final answers only. Download the document with the final answers here.
- Should you need more help, check out the hints and video lecture on the Math Education Resources.

Tips for Using Previous Exams to Study: Exam Simulation

Resist the temptation to read any of the solutions below before completing each question by yourself first! We recommend you follow the guide below.

1. **Exam Simulation:** When you've studied enough that you feel reasonably confident, print the raw exam (click here) without looking at any of the questions right away. Find a quiet space, such as the library, and set a timer for the real length of the exam (usually 2.5 hours). Write the exam as though it is the real deal.
2. **Reflect on your writing:** Generally, reflect on how you wrote the exam. For example, if you were to write it again, what would you do differently? What would you do the same? In what order did you write your solutions? What did you do when you got stuck?
3. **Grade your exam:** Use the solutions in this pdf to grade your exam. Use the point values as shown in the original exam.
4. **Reflect on your solutions:** Now that you have graded the exam, reflect again on your solutions. How did your solutions compare with our solutions? What can you learn from your mistakes?
5. **Plan further studying:** Use your mock exam grades to help determine which content areas to focus on and plan your study time accordingly. Brush up on the topics that need work:
   - Re-do related homework and webwork questions.
   - The Math Education Resources offers mini video lectures on each topic.
   - Work through more previous exam questions thoroughly without using anything that you couldn't use in the real exam. Try to work on each problem until your answer agrees with our final result.
   - Do as many exam simulations as possible.

   Whenever you feel confident enough with a particular topic, move on to topics that need more work. Focus on questions that you find challenging, not on those that are easy for you. Always try to complete each question by yourself first.

This pdf was created for your convenience when you study Math and prepare for your final exams. All the content here, and much more, is freely available on the Math Education Resources.

Question 1 (a)

SOLUTION. The planes are orthogonal if the scalar product of their normal vector is zero.

$$n_Q = (-1, 5, -3)$$
$$n_P = (3, -\frac{9}{5}, -4)$$

$$n_Q \cdot n_P = (-1) \cdot 3 + 5 \cdot (-\frac{9}{5}) + (-3) \cdot (-4) = 0.$$

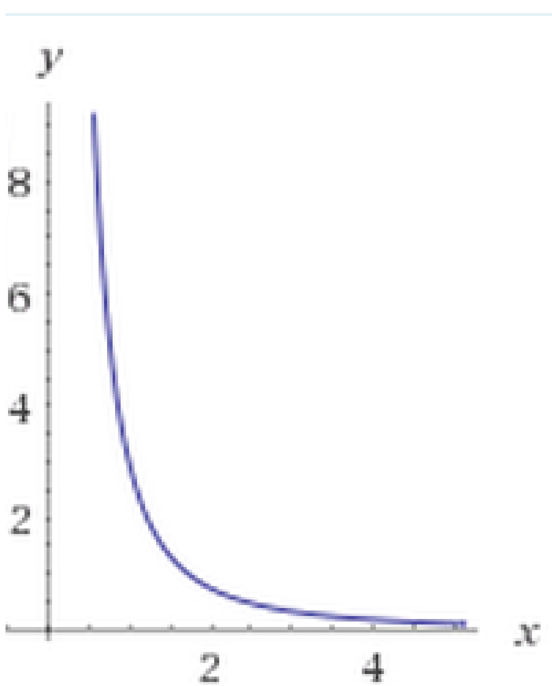
Hence the two planes are orthogonal.

Question 1 (b)

SOLUTION. Setting $V = \pi$,

$$\frac{\pi x^2 y}{3} = \pi$$
$$x^2 y = 3$$
$$y = \frac{3}{x^2}$$

We plot the equation $y = \frac{3}{x^2}$ noting that $x \geq 0$ is necessary for the curve to make physical sense.



Question 1 (c)

SOLUTION. First, take the derivative with respect to y using the chain rule and treating x as a constant. Then use take the derivative with respect to x using the product rule and the chain rule:

$$\begin{aligned}\frac{\partial^2}{\partial x\partial y}f(x,y) &= \frac{\partial^2}{\partial x\partial y}\sin(xy)\\ &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\sin(xy)\right)\\ &= \frac{\partial}{\partial x}\left(\cos(xy)x\right)\\ &= -\sin(xy)xy+\cos(xy)\end{aligned}$$

Question 1 (d)

SOLUTION. Set $a = 5$, $b = 15$ and $n = 50$.

$$\begin{aligned}\Delta x &= \frac{b-a}{n} = \frac{15-5}{50} = \frac{1}{5};\\ x_i &= a + i\Delta x = 5 + \frac{i}{5};\\ \frac{x_{i-1}+x_i}{2} &= \frac{\left(5+\frac{i-1}{5}\right)+\left(5+\frac{i}{5}\right)}{2} = 5 + \frac{2i-1}{10}.\end{aligned}$$

Hence, the Riemann sum is:
$R = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\cdot\Delta x = \sum_{i=1}^{50}\left(5+\frac{2i-1}{10}\right)^8\cdot\frac{1}{5}$.

Question 1 (e)

SOLUTION. Split the interval of integration to $[1, 3]$ and $[3, 5]$:

$$\begin{aligned}\int_1^5 f(x)dx &= \int_1^3 f(x)dx + \int_3^5 f(x)dx\\ &= \int_1^3 3dx + \int_3^5 xdx\\ &= 3x\Big|_1^3 + \frac{x^2}{2}\Big|_3^5\\ &= (9-3) + \left(\frac{25}{2}-\frac{9}{2}\right)\\ &= 14.\end{aligned}$$

Question 1 (f)

SOLUTION 1. We define $G(x)$ as $\frac{1}{2}(f'(x))^2$. Then $G'(x) = \left(\frac{1}{2}(f'(x))^2\right)' = f'(x)f''(x)$ and we have:

$$\begin{aligned}
\int_1^2 f'(x)f''(x)dx &= \int_1^2 G'(x)dx \\
&= G(x)|_1^2 \\
&= \frac{1}{2}(f'(x))^2\Big|_1^2 \\
&= \frac{1}{2}\left(f'(2)\right)^2 - \frac{1}{2}\left(f'(1)\right)^2 \\
&= \frac{9}{2} - 2 \\
&= \frac{5}{2}
\end{aligned}$$

SOLUTION 2. We use integration by parts and set $u = f'(x)$ and $dv = f''(x)dx$. Then
$\int_1^2 f'(x)f''(x)dx = \left[f'(x)^2\right]_1^2 - \int_1^2 f''(x)f'(x)dx$
Note that the integral on the left and right are the same and thus we can rewrite this as
$2\int_1^2 f'(x)f''(x)dx = \left[f'(x)^2\right]_1^2 = 3^2 - 2^2 = 9 - 4 = 5$
Dividing by 2 leads to $\int_1^2 f'(x)f''(x)dx = \frac{5}{2}$

SOLUTION 3. We use integration by substitution. If $u = f'(x)$ then $du = f''(x)dx$. Thus, $f'(x)f''(x)dx = udu$. We also can transform the bounds so that $1 \mapsto f'(1) = 2$ and $2 \mapsto f'(2) = 3$. In terms of u, we can integrate:

$$\int_2^3 udu = \frac{1}{2}u^2|_2^3 = \frac{5}{2}.$$

This is our final answer.

Question 1 (g)

SOLUTION. Recall the formula for integration by parts

$$\int udv = uv - \int vdu.$$

Let $u = \cos^{-1}(y)$ and $dv = dy$. Then $du = \frac{-1}{\sqrt{1-y^2}}dy$ and $v = y$.

$$\begin{aligned}
\int \cos^{-1}(y)dy &= y\cos^{-1}(y) - \int y(\cos^{-1}(y))'dy \\
&= y\cos^{-1}(y) + \int \frac{y}{\sqrt{1-y^2}}dy
\end{aligned}$$

Now, recognizing y as the derivative of $1 - y^2$ (to within a factor of -1/2), we can do a substitution with $x = 1 - y^2$ and $dx = -2ydy$.

$$\begin{aligned}
\int \cos^{-1}(y)dy &= y\cos^{-1}(y) - \frac{1}{2}\int \frac{1}{\sqrt{x}}dx \\
&= y\cos^{-1}(y) - \sqrt{x} + C \\
&= y\cos^{-1}(y) - \sqrt{1-y^2} + C
\end{aligned}$$

where we computed $\int \frac{1}{\sqrt{x}}dx = \int x^{-1/2}dx = \frac{1}{-1/2+1}x^{-1/2+1} + C = 2\sqrt{x} + C$ by the reverse power rule.

Question 1 (h)

SOLUTION. Following the remark in the hint, we begin by factoring out $\cos x$, the derivative of $\sin x$ so that we can later do a substitution. In doing this substitution, we will need to express the integrand in terms of powers of $\sin x$ with a single factor of $\cos x$.

$$\begin{aligned}\int \cos^3(x)\sin^4(x)dx &= \int \cos(x)\cos^2(x)\sin^4(x)dx \\ &= \int \cos(x)(1-\sin^2(x))\sin^4(x)dx \\ &= \int (\sin^4(x)-\sin^6(x))\cos(x)dx\end{aligned}$$

With $u = \sin(x)$, $du = \cos(x)dx$ giving

$$\begin{aligned}\int (u^4 - u^6)du &= \frac{u^5}{5} - \frac{u^7}{7} + C \\ &= \frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C.\end{aligned}$$

Question 1 (i)

SOLUTION. First we complete the square in the expression $3-2x-x^2$ by considering $3-2x-x^2 = 3-(x^2+2x) = 3-((x+1)^2-1) = 4-(x+1)^2$.
This changes the integral to $\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{dx}{\sqrt{4-(x+1)^2}}$
We substitute $t = x+1$ with $dx = dt$ and obtain $\int \frac{dt}{\sqrt{4-t^2}}$
We do another substitution with $t = 2\sin(u)$ (so that $u = \sin^{-1}(\frac{t}{2})$) and $dt = 2\cos(u)du$ giving

$$\begin{aligned}\int \frac{2\cos(u)du}{\sqrt{4-4\sin(u)^2}} &= \int \frac{2\cos(u)du}{2\sqrt{1-\sin(u)^2}} \\ &= \int \frac{\cos(u)du}{\sqrt{\cos(u)^2}}\end{aligned}$$

where we used that $1-\sin(u)^2 = \cos(u)^2$.
The remaining integral is $\int \frac{\cos(u)}{\cos(u)}du = \int du = u + C$.
Now we re-substitute $u + C = \sin^{-1}\left(\frac{t}{2}\right) + C = \sin^{-1}\left(\frac{x+1}{2}\right) + C$.

Question 1 (j)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Factor the denominator:
$x^2 - x - 6 = (x-3)(x+2)$
Use partial fraction:

$$\begin{aligned}\frac{x-13}{x^2-x-6} &= \frac{A}{x-3}+\frac{B}{x+2}\\ &= \frac{Ax+2A+Bx-3B}{(x-3)(x+2)}\\ &= \frac{(A+B)x+(2A-3B)}{(x-3)(x+2)}\end{aligned}$$

Comparing the coefficients we get:

$$\begin{aligned}A+B&=1\\ 2A-3B&=-13\end{aligned}$$

Solve and get:

$$\begin{aligned}A&=-2\\ B&=3\end{aligned}$$

Hence:

$$\begin{aligned}\int \frac{x-13}{x^2-x-6}dx &= \int \frac{-2}{x-3}+\frac{3}{x+2}dx\\ &= -2\ln|x-3|+3\ln|x+2|+C\end{aligned}$$

Question 1 (k)

SOLUTION.

$$\begin{aligned}\mathbb{E}(X) &= \int_{-\infty}^{\infty} xf(x)dx\\ &= \int_{-\infty}^{1} xf(x)dx + \int_{1}^{\infty} xf(x)dx\\ &= \int_{-\infty}^{1} 0dx + \int_{1}^{\infty} x\cdot\frac{3}{2}x^{-\frac{5}{2}}dx\\ &= 0+\int_{1}^{\infty}\frac{3}{2}x^{-\frac{3}{2}}dx\\ &= \lim_{b\to\infty}\int_{1}^{b}\frac{3}{2}x^{-\frac{3}{2}}dx\\ &= \lim_{b\to\infty} -3x^{-\frac{1}{2}}\Big|_{1}^{b}\\ &= \lim_{b\to\infty} -3b^{-\frac{1}{2}}-(-3(1)^{-\frac{1}{2}})\\ &= 3.\end{aligned}$$

Question 1 (l)

SOLUTION. We have

$$\begin{aligned}\sum_{n=1}^{\infty}\left[\left(\frac{1}{3}\right)^n+\left(-\frac{2}{5}\right)^{n-1}\right] &= \sum_{n=0}^{\infty}\left[\left(\frac{1}{3}\right)^{n+1}+\left(-\frac{2}{5}\right)^{n}\right]\\ &= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n+\sum_{n=0}^{\infty}\left(-\frac{2}{5}\right)^n\\ &= \frac{1}{3}\frac{1}{1-1/3}+\frac{1}{1-(-2/5)}\\ &= \frac{1}{2}+\frac{5}{7}\\ &= \frac{17}{14}.\end{aligned}$$

With the first equality, we rewrote the geometries series starting from $n = 0$ in order to apply $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ for $|r| < 1$. This was accomplished by re-indexing. In the second equality, we split the sum into two and factored out $1/3$ from the first sum.

Question 1 (m)

SOLUTION 1. We must have that $\int_{-1}^{1} f(x)dx = 1$ because $f(x)$ is a probability density function (which is zero outside of [-1,1]). Hence, we have:

$$\begin{aligned}1 &= \int_{-1}^{1}(1+k|x|)dx\\ &= 2\int_{0}^{1}(1+k|x|)dx\\ &= 2\int_{0}^{1}(1+kx)dx\\ &= 2\left(x+\frac{kx^2}{2}\right)\Big|_0^1\\ &= 2\left(1+\frac{k}{2}\right)\\ &= 2+k.\end{aligned}$$

Note that in the second equality we used that $\displaystyle\int_{-1}^{1} f(x)dx = 2\int_{0}^{1} f(x)dx$ since $f(x)$ is an even function; i. e. $f(x) = f(-x)$. In the third equality, we have used that for $x > 0$, $|x| = x$.
Finally we get $1 = 2 + k$. Solve it to get $k = -1$.

SOLUTION 2. We must have that $\int_{-1}^{1} f(x)dx = 1$ since $f(x)$ is a probability density function which is zero outside of [-1,1]. Hence, we have:

$$\begin{aligned}1 &= \int_{-1}^{1}(1+k|x|)dx\\ &= \int_{-1}^{0}(1+k|x|)dx+\int_{0}^{1}(1+k|x|)dx\end{aligned}$$

Now, we use that that

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

to write

$$\begin{aligned} 1 &= \int_{-1}^{0} (1 - kx)dx + \int_{0}^{1} (1 + kx)dx \\ &= \left(x - \frac{kx^2}{2} \right) \Big|_{-1}^{0} + \left(x + \frac{kx^2}{2} \right) \Big|_{0}^{1} \\ &= -\left(-1 - \frac{k}{2} \right) + \left(1 + \frac{k}{2} \right) \\ &= 2 + k. \end{aligned}$$

Solving $1 = 2 + k$ gives us $k = -1$.

Question 1 (n)

SOLUTION. **THIS QUESTION HAS NOT YET BEEN REVIEWED! THE SOLUTION BELOW MAY CONTAIN MISTAKES!**

Let $t = xy$ and $G(t) = \int_1^t h(s)ds$. Then $G'(t) = h(t)$. By chain rule, we get:
$f_x(x, y) = G'(t)\frac{dt}{dx} = G'(xy)y = h(xy)y$.
Hence, $f_x(2, 5) = h(2 \cdot 5) \cdot 5 = h(10) \cdot 5 = 2 \cdot 5 = 10$.

Question 2 (a)

SOLUTION. Since $\dfrac{\sqrt[3]{k^4 + 1}}{\sqrt{k^5 + 9}} \approx \dfrac{\sqrt[3]{k^4}}{\sqrt{k^5}} = \dfrac{k^{4/3}}{k^{5/2}} = \dfrac{1}{k^{5/2-4/3}} = \dfrac{1}{k^{7/6}}$, we want to compare $\dfrac{\sqrt[3]{k^4 + 1}}{\sqrt{k^5 + 9}}$ with $\dfrac{1}{k^{7/6}}$.
Let $a_k = \dfrac{1}{k^{7/6}} > 0$ and $b_k = \dfrac{\sqrt[3]{k^4 + 1}}{\sqrt{k^5 + 9}} > 0$. Recall $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$ converges if and only if $\alpha > 1$. Hence $\displaystyle\sum_{k=1}^{\infty} a_k$ converges.
On the other hand, by the Limit Comparison Test, if $\displaystyle\lim_{k\to\infty} \frac{b_k}{a_k} = L$ for some nonzero finite value of L and both series are positive then either both $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=1}^{\infty} b_k$ converge or both diverge. Hence, if we can prove $\displaystyle\lim_{k\to\infty} \frac{b_k}{a_k} = 1$, then we know that $\displaystyle\sum_{k=1}^{\infty} b_k$ converges. They key fact here is that we know the convergence properties of one of the two series we are comparing, namely the series with $\frac{1}{k^{7/6}}$.

$$\lim_{k\to\infty}\frac{b_k}{a_k} = \lim_{k\to\infty} k^{7/6}\frac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}} = \lim_{k\to\infty} k^{7/6}\frac{\sqrt[3]{(1+k^{-4})\cdot k^4}}{\sqrt{(1+9k^{-5})\cdot k^5}}$$

$$= \lim_{k\to\infty} k^{7/6}\frac{\sqrt[3]{(1+k^{-4})}}{\sqrt{(1+9k^{-5})}}\frac{k^{4/3}}{k^{5/2}} = \lim_{k\to\infty} k^{7/6}\frac{\sqrt[3]{(1+k^{-4})}}{\sqrt{(1+9k^{-5})}}k^{4/3-5/2}$$

$$= \lim_{k\to\infty} k^{7/6}\frac{\sqrt[3]{(1+k^{-4})}}{\sqrt{(1+9k^{-5})}}k^{-7/6} = \lim_{k\to\infty}\frac{\sqrt[3]{(1+k^{-4})}}{\sqrt{(1+9k^{-5})}}$$

$$= \lim_{k\to\infty}\frac{\sqrt[3]{(1+k^{-4})}}{\sqrt{(1+9k^{-5})}} = \frac{1}{1} = 1.$$

Question 2 (b)

SOLUTION. Let $a_k = \frac{x^k}{10^{k+1}(k+1)!}$ be the terms in the series. From the ratio test, we are guaranteed absolute convergence when $\lim_{k\to\infty}|\frac{a_{k+1}}{a_k}| < 1$.
With $a_k = \frac{x^k}{10^{k+1}(k+1)!}$, we have $a_{k+1} = \frac{x^{k+1}}{10^{k+2}(k+2)!}$ and thus

$$\begin{aligned}|\frac{a_{k+1}}{a_k}| &= |\frac{\frac{x^{k+1}}{10^{k+2}(k+2)!}}{\frac{x^k}{10^{k+1}(k+1)!}}| \\ &= |\frac{x^{k+1}10^{k+1}(k+1)!}{x^k 10^{k+2}(k+2)!}| \\ &= |\frac{x}{10}\frac{(k+1)!}{(k+2)!}|.\end{aligned}$$

We recall that $k! = k(k-1)(k-2)...(3)(2)(1)$ and thus in general $k! = k(k-1)!$. This also means that $(k+2)! = (k+2)(k+1)!$. Thus,

$$\begin{aligned}|\frac{a_{k+1}}{a_k}| &= |\frac{x(k+1)!}{10(k+2)(k+1)!}| \\ &= |\frac{x}{10(k+2)}|\end{aligned}$$

.
Computing $\lim_{k\to\infty}|\frac{x}{10(k+2)}| = 0 < 1$ for all x. Hence, the series converges absolutely for all x-values and the radius of convergence is ∞.

Question 2 (c)

SOLUTION. Using $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$, we have
$\frac{3}{x+1} = \frac{3}{1-(-x)} = 3\cdot\sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}3(-1)^n x^n$;
$-\frac{1}{2x-1} = \frac{1}{1-2x} = \sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty}2^n x^n$;
Therefore,

$$\frac{3}{x+1} - \frac{1}{2x-1} = \sum_{n=0}^{\infty}3(-1)^n x^n + \sum_{n=0}^{\infty}2^n x^n = \sum_{n=0}^{\infty}(3(-1)^n + 2^n)x^n.$$

Hence, $b_n = 3(-1)^n + 2^n$.

Question 3 (a)

SOLUTION. The object function is $f(x, y) = (x + 1)^2 + (y - 2)^2$ and the constraint is $x^2 + y^2 = 125$, i. e. $g(x, y) = x^2 + y^2 - 125 = 0$.
By the method Lagrange multipliers, set $\nabla f = \lambda \nabla g, g = 0$ which tells us $\langle 2(x+1), 2(y-2)\rangle = \lambda\langle 2x, 2y\rangle, \quad x^2 + y^2 - 125 = 0$.
Looking at the vector equation in components, we have that

$$2(x + 1) = 2\lambda x \implies x + 1 = \lambda x \quad (1)$$
$$2(y - 1) = 2\lambda y \implies y - 2 = \lambda y. \quad (2)$$

Provided we don't divide by zero (so that $y \neq 2$), we can divide (1) by (2) to yield $\frac{x+1}{y-2} = \frac{\lambda x}{\lambda y} = \frac{x}{y} \implies xy + y = xy - 2x \implies y = -2x$ where the first implication came by cross multiplying.
If $y = x$ then from the constraint $x^2 + y^2 - 125 = 0$ we must have $x^2 + (-2x)^2 - 125 = 5x^2 - 125 = 0 \implies x = \pm 5$ and $y = -2x = \mp 10$.
Evaluating $f(5, -10) = (5 + 1)^2 + (-10 + 2)^2 = 180$ and $f(-5, 10) = (-5 + 1)^2 + (10 - 2)^2 = 80$.
We still need to consider the possibility that $y = 2$. If $y = 2$ then (2) reads $0 = 2\lambda \implies \lambda = 0$. If $\lambda = 0$ then (1) tells us that $x + 1 = 0$ so that $x = -1$. However, $(-1, 2)$ does not satisfy $g(x, y) = 0$ so this is not a valid solution to the Lagrange system.
Overall have found that the maximum value is 180 and the minimum value is 80.

Question 3 (b)

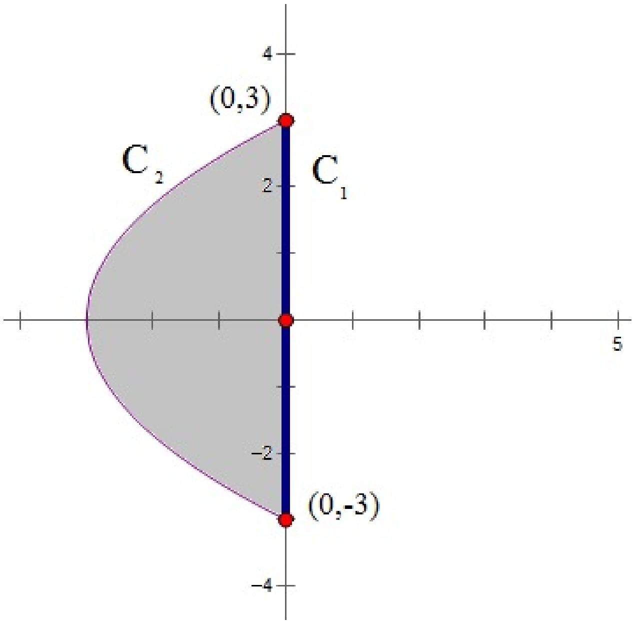
SOLUTION. We want to find where $\sqrt{(x - (-1))^2 + (y - 2)^2}$ attains its minimum on the circle $x^2+y^2 = 125$. Equivalently, we need to find where $(x - (-1))^2 + (y - 2)^2 = (x + 1)^2 + (y - 2)^2$ is the smallest on the circle $x^2 + y^2 = 125$. From (a), we know that the minimum is at $(-5, 10)$.

Question 4 (a)

SOLUTION. Compute the derivatives:
$T_x(x, y) = 2x - 2y + 6 \qquad T_y(x, y) = \frac{1}{3}y^2 - 2x - 6$;
$T_{xx}(x, y) = 2, \quad T_{xy}(x, y) = -2, \quad T_{yy}(x, y) = \frac{2}{3}y$.
Set $T_x(x, y) = 0$ and $T_y(x, y) = 0$ to find critical points:
$2x - 2y + 6 = 0$
$\frac{1}{3}y^2 - 2x - 6 = 0$.
From the first equality we get $x = y - 3$. Plugging this into the second equality, we get $\frac{1}{3}y^2 - 2y = 0$. Solving it gives $y = 0$ and $y = 6$. This yields the critical points $(x, y) = (-3, 0)$, and $(x, y) = (3, 6)$.
For $(-3, 0)$, $T_{xx} = 2$, $T_{yy} = \frac{2}{3}y = 0$ and $T_{xy} = -2$, so we have a saddle point because $T_{xx}T_{yy} - T_{xy}^2 = -4 < 0$.
For $(3, 6)$, $T_{xx} = 2$, $T_{yy} = \frac{2}{3}y = 4$, and $T_{xx} \cdot T_{yy} - T_{xy}^2 = 2 \cdot \frac{2}{3}y - (-2)^2 = 2 \cdot 4 - (-2)^2 = 4 > 0$ (and thus it could be a local max or local min). Then, because $T_{xx} = 2 > 0$, we conclude it is a local minimum.

Question 4 (b)

SOLUTION. We just need to check the local maximum, the local minimum and the points on the boundary. From question 4a above, we know that it has only one local minimum $(3, 6)$ and no local maximum. Since $(3, 6)$ is not on R, we only need to find the min and max on the boundary.



On C_1:
$C_1 = \{x = 0, y \in [-3, 3]\}$,

$$T(x, y) = \frac{1}{9}y^3 - 6y = f(y) \qquad f'(y) = \frac{1}{3}y^2 - 6$$

The critical points from above occur when $\frac{1}{3}y^2 - 6 = 0$ which is at $y = \pm\sqrt{18} = \pm 3\sqrt{2}$, neither of which are in $[-3, 3]$. Thus, we just test the endpoints

$$f(-3) = 15$$

and $f(3) = -15$. These are two values we will consider later.
On C_2:
$C_2 = \{x = \frac{1}{3}y^2 - 3, y \in [-3, 3]\}$,
$T(x, y) = \frac{1}{9}y^3 + \left(\frac{1}{3}y^2 - 3\right)^2 - 2\left(\frac{1}{3}y^2 - 3\right)y + 6\left(\frac{1}{3}y^2 - 3\right) - 6y = \frac{y^4}{9} - \frac{5}{9}y^3 - 9 = g(y)$
$g'(y) = \frac{4}{9}y^3 - \frac{15}{9}y^2 = \frac{y^2}{9}(4y - 15) = 0$ at $y = 0$ and $y = 15/4$. As $15/4$ is not in $[-3, 3]$ we ignore it. We now compute $g(0) = -9$ and the value at the endpoints with $g(-3) = 15$ and $g(3) = -15$.
From all of this, we find max $= 15$ and min $= -15$.

Question 5 (a)

SOLUTION. We first solve for the differential equation:

$$\begin{aligned}
\frac{dB}{dt} &= aB - m \\
dB &= (aB - m)dt \\
\frac{dB}{aB - m} &= dt \quad \text{(separating the variables)} \\
\int \frac{dB}{aB - m} &= \int dt \\
\frac{1}{a}\ln|aB - m| &= t + C \\
\ln|aB - m| &= at + C \quad \text{(or aC, but C is arbitrary so we still call it C)} \\
|aB - m| &= e^{at+C} = Ae^{at} \quad \text{(where } A = e^C\text{)} \\
aB - m &= \pm Ae^{at} \quad \text{(removing the absolute values gives a } \pm\text{)} \\
B &= \frac{1}{a}(m - Ae^{at}) \quad \text{(A here is } \pm A \text{ is arbitrary and still unknown)}
\end{aligned}$$

Now we can use the initial conditions with $B(0) = 30000$ and $a = 0.02 = 1/50$ to find:
$B(0) = 30000 = 50(m - A) \implies A = m - 600$ and thus
$B(t) = 50(m - (m - 600)e^{0.02t}) = 50((600 - m)e^{0.02t} + m)$.

Question 5 (b)

SOLUTION 1. From $B(t) = 50((600 - m)e^{0.02t} + m)$, we can see that if $600 - m = 0$, then $B(t)$ is a constant. Hence, $m = 600$.

SOLUTION 2. If $B(t)$ is a constant then $B'(t)$ must be 0. Therefore

$$\begin{aligned}
0 &= B'(0) \\
&= aB(0) - m \\
&= 0.02 \cdot 30000 - m \\
&= 600 - m
\end{aligned}$$

Hence, $m = 600$.

Question 6 (a)

SOLUTION. Since $e^x = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}$, let $x = \frac{1}{\pi}$ and we get
$e^{\frac{1}{\pi}} = 1 + \sum_{k=1}^{\infty} \frac{(\frac{1}{\pi})^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{1}{\pi^k k!}$
Hence
$\sum_{k=1}^{\infty} \frac{1}{\pi^k k!} = e^{\frac{1}{\pi}} - 1$

Question 6 (b)

SOLUTION. Since $\sum_{n=1}^{\infty} \frac{na_n - 2n + 1}{n + 1}$ converges, we have $\lim_{n\to\infty} \frac{na_n - 2n + 1}{n + 1} = 0$. Hence:

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{na_n - 2n + 1}{n+1} \\
&= \lim_{n\to\infty} \left((a_n - 2) \cdot \frac{n}{n+1} + \frac{1}{n+1} \right) \\
&= \lim_{n\to\infty} (a_n - 2) \cdot \frac{n}{n+1} + \lim_{n\to\infty} \frac{1}{n+1} \\
&= \lim_{n\to\infty} (a_n - 2) \cdot \frac{n}{n+1} + 0 \\
&= \lim_{n\to\infty} (a_n - 2) \cdot \lim_{n\to\infty} \frac{n}{n+1} \\
&= \lim_{n\to\infty} (a_n - 2) \cdot 1 \\
&= \lim_{n\to\infty} (a_n - 2)
\end{aligned}$$

Therefore $\lim_{n\to\infty} (a_n - 2) = 0$. Or rather, $\lim_{n\to\infty} a_n = 2$.
On the other hand, $\ln\left(\frac{a_n}{a_{n+1}}\right) = \ln a_n - \ln a_{n+1}$. Hence,

$$\begin{aligned}
-\ln a_1 + \sum_{n=1}^{k} \ln\left(\frac{a_n}{a_{n+1}}\right) &= -\ln a_1 + \sum_{n=1}^{k} (\ln a_n - \ln a_{n+1}) \\
&= -\ln a_1 + (\ln a_1 - \ln a_2) + (\ln a_2 - \ln a_3) + \cdots + (\ln a_k - \ln a_{k+1}) \\
&= -\ln a_{k+1}
\end{aligned}$$

Hence,

$$\begin{aligned}
-\ln a_1 + \sum_{n=1}^{\infty} \ln\left(\frac{a_n}{a_{n+1}}\right) &= \lim_{k\to\infty} \left(-\ln a_1 + \sum_{n=1}^{k} \ln\left(\frac{a_n}{a_{n+1}}\right) \right) \\
&= \lim_{k\to\infty} -\ln a_{k+1} \\
&= -\ln 2.
\end{aligned}$$

Good Luck for your exams!