Welcome to Math 105

My name is Elyse Yeager.

Course webpage: www.math.ubc.ca/~kliu/common105.html

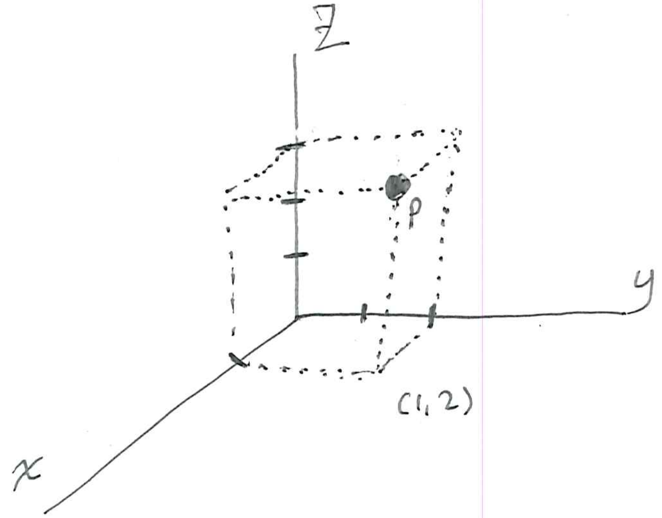
Section webpage: www.math.ubc.ca/~elyse/2017Math105.html

Book: Briggs, Cochran, Gillet
Calculus: Early Transcendentals
Volume 2
Fourth custom edition for UBC

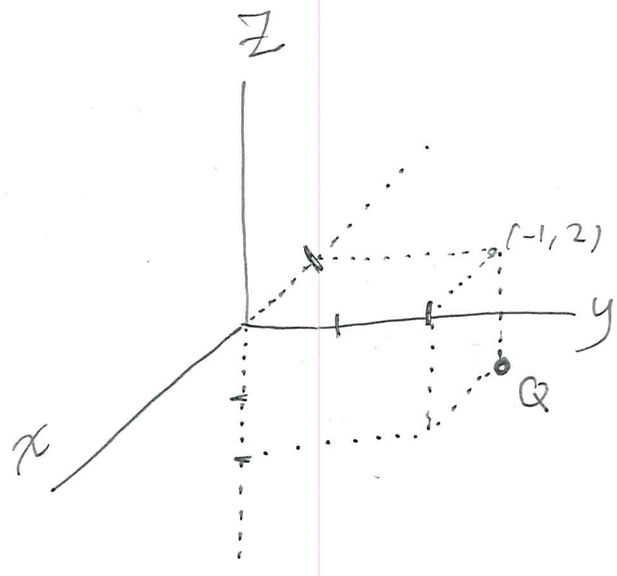
webwork.elearning.ubc.ca

Ch. 11.1–11.3 Vectors & Dot Products

Drawing in $\mathbb{R}^3$ ← xyz coordinates



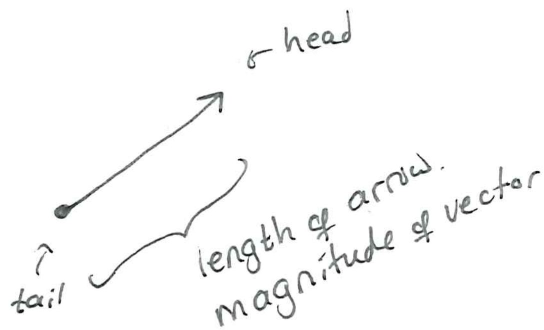
$P = (1, 2, 3)$

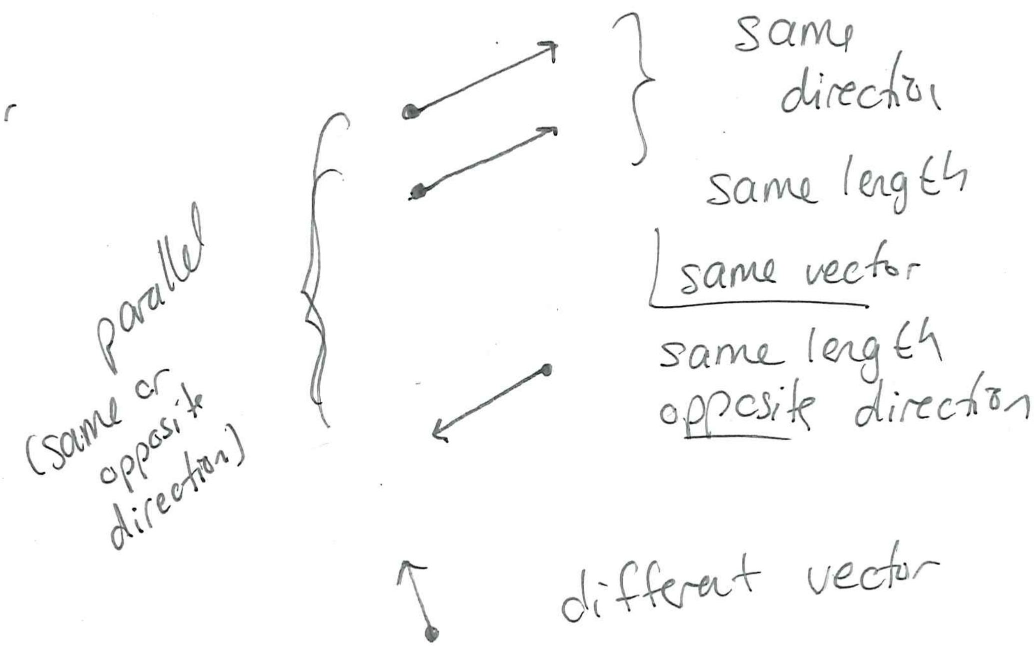


$Q = (-1, 2, -2)$

Vectors

A vector is a mathematical object with a magnitude (length, size) and a direction.

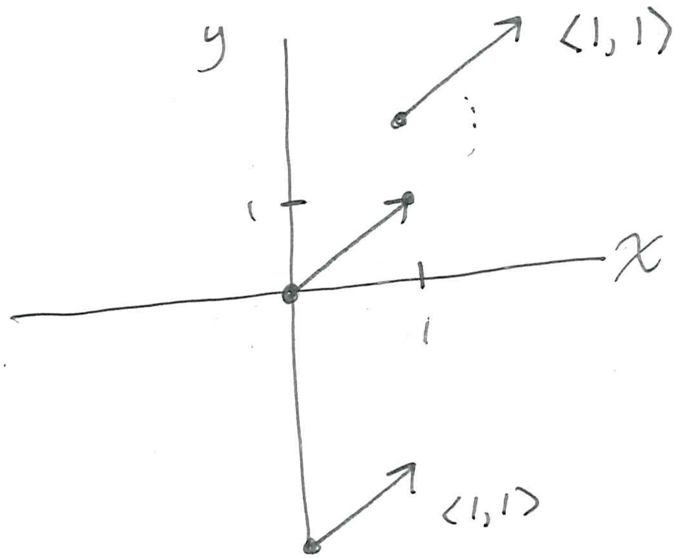


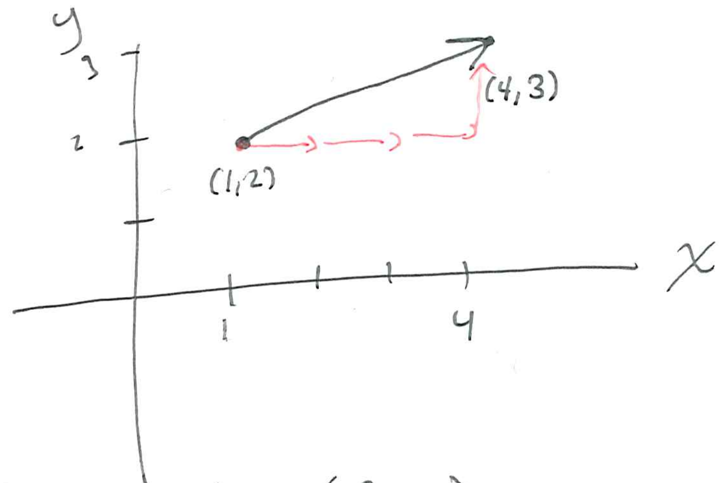


Naming Vectors

In a coordinat system (e.g. $\underbrace{\text{xy-plane}}_{\mathbb{R}^2}$, $\underbrace{xyz}_{\mathbb{R}^3}$)

we name a vector by
the position of its head when its tail
is at origin.



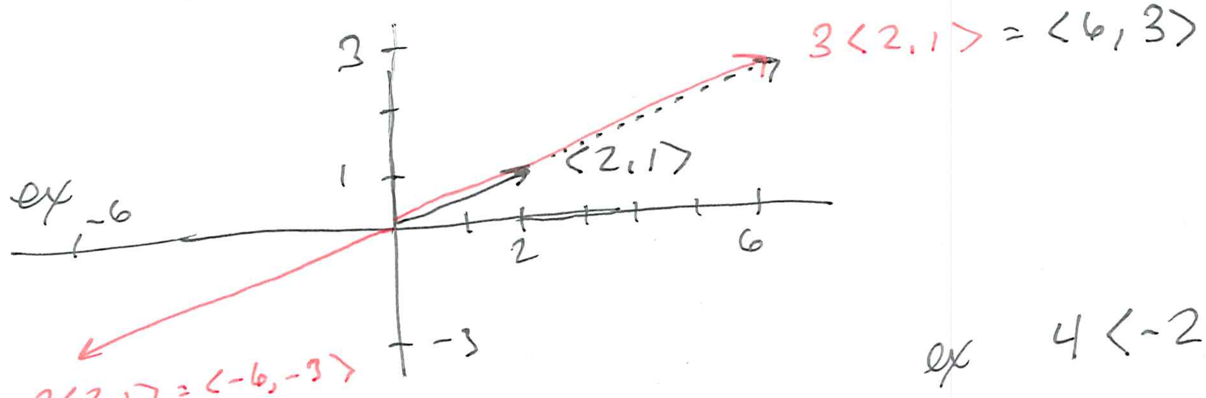


⟨3, 1⟩

Multiplying a Vector & a $\underbrace{\text{Scalar}}_{\text{number}}$

When we multiply a vector by a positive scalar,
we leave the direction alone, and multiply the length

ex



ex $4\langle -2, 1, 0\rangle =$

$\langle -8, 4, 0\rangle$

When we multiply a vector by a negative scalar,
we get a vector in the opposite direction (swap head & tail)
& multiply length

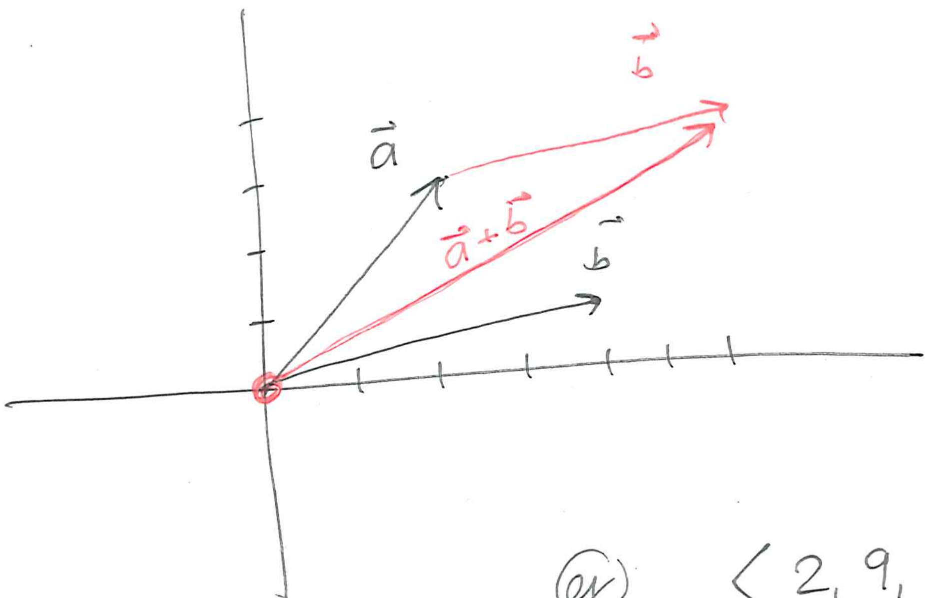
- Vectors that are parallel are scalar multiples of one another

Adding Vectors

If we add vectors $\vec{a}$ and $\vec{b}$:

we put them head-to-tail

$\vec{a}+\vec{b}$ is the vector w/ tail @ 1st tail
head @ 2nd head



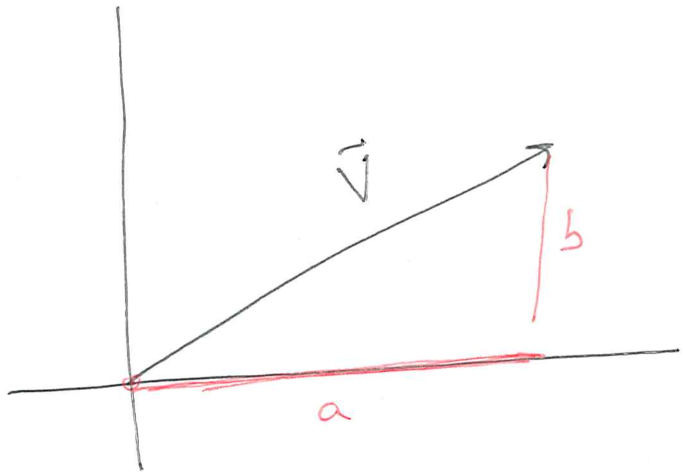
$\vec{a} = \langle 2, 3 \rangle$

$\vec{b} = \langle 4, 1 \rangle$

$\vec{a}+\vec{b} = \langle 6, 4 \rangle$

ex $\langle 2, 9, 20 \rangle + \langle -4, 8, 6 \rangle = \langle -2, 17, 26 \rangle$

Length + Direction of Vectors



$\vec{v} = \langle a, b \rangle$

$\underbrace{\| \vec{v} \|}_{\text{"length" or "norm" or "magnitude" of } \vec{v}} = \sqrt{a^2+b^2}$ (Pythagorean Thm)

$\vec{w} = \langle a, b, c \rangle$

Then $\underbrace{\| \vec{w} \|}_{\text{length}} = \sqrt{a^2+b^2+c^2}$

ex What is the length of $\vec{w} = \langle 2, 5, -1 \rangle$?

$$\sqrt{4+25+1} = \boxed{\sqrt{30}}$$

<u>unit vector</u> : any vector of length one

we use these to describe direction.

Compute: $\vec{u}$ unit vector in same direction as $\vec{w}$.

$$\vec{u} = \frac{1}{\sqrt{30}} \langle 2, 5, -1 \rangle = \boxed{\left\langle \frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{-1}{\sqrt{30}} \right\rangle}$$

Check that $\|\vec{u}\| = 1$:

$$\|\vec{u}\| = \sqrt{\frac{4}{30} + \frac{25}{30} + \frac{1}{30}} = \sqrt{\frac{30}{30}} = \sqrt{1} = 1 \checkmark$$

ex Find a vector of length ℓ $(\ell > 0)$
in the same direction as $\langle a, b, c \rangle$.
($\langle a, b, c \rangle$ not all 0s)

$$\| \langle a, b, c \rangle \| = \sqrt{a^2+b^2+c^2}$$

vector: $\dfrac{\ell}{\sqrt{a^2+b^2+c^2}} \langle a, b, c \rangle = \boxed{\left\langle \dfrac{\ell a}{\sqrt{a^2+b^2+c^2}}, \dfrac{\ell b}{\sqrt{a^2+b^2+c^2}}, \dfrac{\ell c}{\sqrt{a^2+b^2+c^2}} \right\rangle}$

has length ℓ;
same direction as $\langle a, b, c \rangle$

Dot Product

The dot product is calculated like this:

(in $\mathbb{R}^2$) $\qquad \underbrace{\langle a, b \rangle}_{\text{vector}} \cdot \underbrace{\langle x, y \rangle}_{\text{vector}} = \underbrace{ax + by}_{\substack{\text{number} \\ \text{(scalar)}}}$

(in $\mathbb{R}^3$) $\qquad \langle a, b, c \rangle \cdot \langle x, y, z \rangle = ax + by + cz$

ex $\qquad \langle 2, 5, -1 \rangle \cdot \langle 3, -2, 0 \rangle = 6 - 10 + 0 = -4$

$\boxed{\text{FACT}}$ If $\vec{u}$ and $\vec{v}$ are perpendicular (orthogonal), then
$\vec{u} \cdot \vec{v} = 0$.

ex: $\langle 2, 5, -1 \rangle$, $\langle 3, -2, 0 \rangle$ <u>not</u> perpendicular (ex) $\vec{a} = \langle 1, 0, 3 \rangle$

$\vec{b} = \langle 3, 0, -1 \rangle$

$\vec{c} = \langle -2, 0, -6 \rangle$

Which pairs are parallel? $(-2)\vec{a} = \vec{c}$ so $\boxed{\vec{a}, \vec{c} \text{ parallel}}$

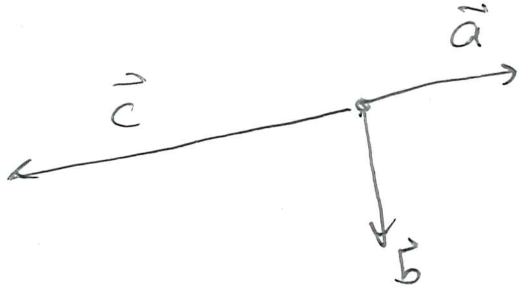
Which are perpendicular?

$\vec{a} \cdot \vec{b} = 3 + 0 - 3 = 0$ so $\boxed{\vec{a}, \vec{b} \text{ perpendicular}}$

$\vec{b} \cdot \vec{c} = -6 + 0 + 6 = 0$ so also $\boxed{\vec{b}, \vec{c} \text{ perpendicula}}$

$\perp$

$\parallel$



rough idea

Properties of Dot Product

$\vec{a}, \vec{b}, \vec{c}$ vectors

s scalar

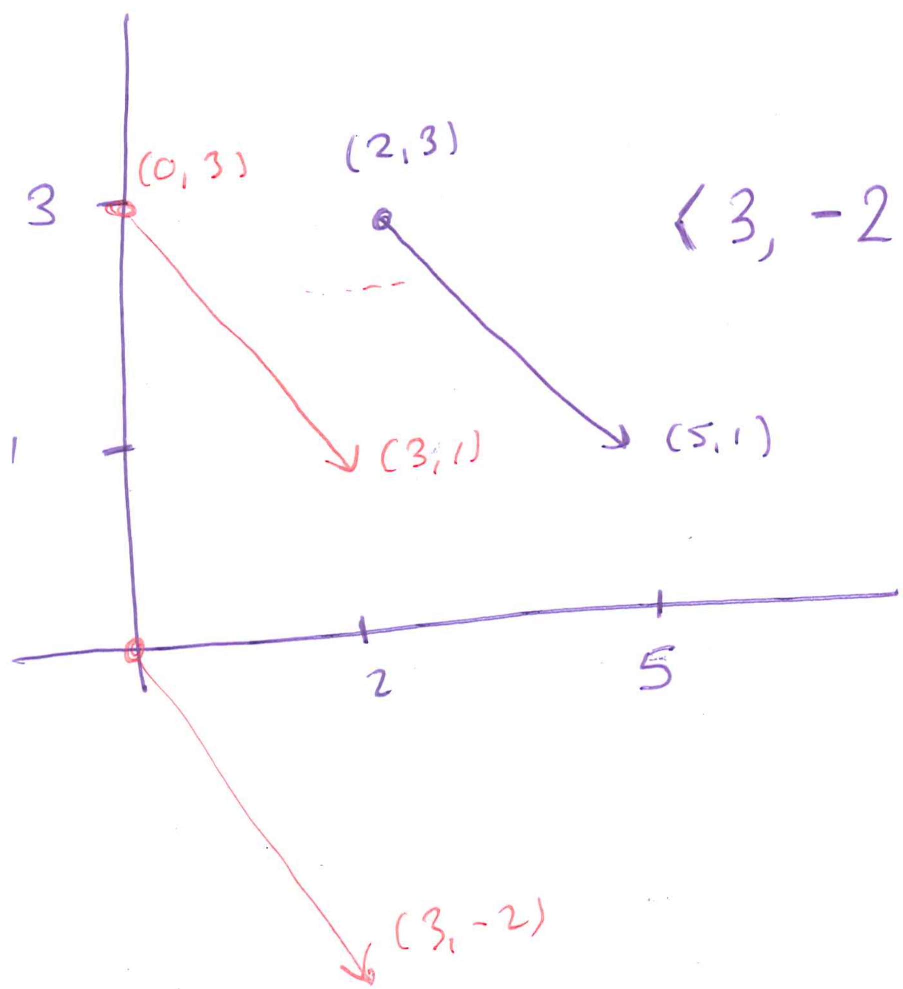
- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $s(\vec{a} \cdot \vec{b}) = (s\vec{a}) \cdot \vec{b}$
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

Suggested Problems

section webpage

$\boxed{11.1 - 11.3}$

Last Time:
Vectors



$\langle 3, -2 \rangle$

$\langle 1, 2 \rangle + \langle 7, 8 \rangle =$

$\langle 8, 10 \rangle$

$-3 \langle 1, 2 \rangle = \langle -3, -6 \rangle$

Ch. 12.1 : Planes + Surfaces

Def: Given a fixed point P_0 and a nonzero vector $\vec{n}$ in $\underbrace{\mathbb{R}^3}$, → (xyz system / 3 dimensions)

the set of points P in $\mathbb{R}^3$ such that $\overrightarrow{P_0P}$ is perpendicular (orthogonal) to $\vec{n}$ is called a plane.

The vector $\vec{n}$ we call a normal vector to P.

Find equation for plane P that passes through point (p_1, p_2, p_3) and has normal vector $\langle n_1, n_2, n_3 \rangle$

Def $\Rightarrow$ P is all points (x, y, z) such that the vector from (p_1, p_2, p_3) to (x, y, z) $\Big\}$ $\langle x-p_1, y-p_2, z-p_3 \rangle$ is perp. to $\langle n_1, n_2, n_3 \rangle$.

That is: $\langle n_1, n_2, n_3 \rangle \cdot \langle x-p_1, y-p_2, z-p_3 \rangle = 0$

$$n_1(x-p_1) + n_2(y-p_2) + n_3(z-p_3) = 0$$

$$n_1 x - n_1 p_1 + n_2 y - n_2 p_2 + n_3 z - n_3 p_3 = 0$$

$$\boxed{n_1 x + n_2 y + n_3 z = n_1 p_1 + n_2 p_2 + n_3 p_3}$$

Suppose a plane P has normal vector

$\vec{n} = \langle 1, 2, 1 \rangle$.

Also: $\langle -1, -2, -1 \rangle$ is a normal vector to P

$\langle 3, 6, 3 \rangle$

$\langle -4, -8, -4 \rangle$

$\langle \pi, 2\pi, \pi \rangle$

ex What is the equation for a plane w/ normal vector $\langle 2, 7, -19 \rangle$ passing through point $(1, -1, 0)$?

$$2x + 7y - 19z = 2 - 7 + 0 = -5$$

$$\boxed{2x + 7y - 19z = -5}$$

(ex) P is a plane

$\vec{n} = \langle 5, 18, 0 \rangle$

has point $P_0 = (3, 0, 9)$

Equation of P:

$5x + 18y = 15$

Q: Is the point $(0, 1, 3)$ on plane P?

Q: For what value of b is $(1, b, 7)$ on the plane P?

Point $(0, 1, 3)$: $5x + 18y = 5 \cdot 0 + 18 \cdot 1 = 18 \neq 15$

not on plane

Point $(1, b, 7)$: $5(1) + 18(b) = 15$

$18b = 10$

$b = \frac{10}{18}$ $\boxed{b = \frac{5}{9}}$

ex Plane has equation:

$$14x - 32y + \pi z = 521$$

1. give a normal vector to plane $\langle 14, -32, \pi \rangle$

2. give me a point on the plane

e.g. $\left(0, 0, \frac{521}{\pi}\right)$

$\left(\frac{521}{14}, 0, 0\right)$

Parallel & Orthogonal Planes

perpendicular
right angles

Parallel planes have parallel normal vectors
Perpendicular planes have perpendicular normal vectors

ex Give a plane parallel to

$3x - 2y + z = 0$

passing through the point

$(1, 1, 1)$

use $\vec{n}$ parallel to $\langle 3, -2, 1 \rangle$

use $\langle 3, -2, 1 \rangle$

$$\boxed{3x - 2y + z = 2}$$

ex Find $\vec{v}$ that is perpendicular to $\langle 3, -2, 1 \rangle$ ($\vec{v}$ not all 0s)

$\langle 2, 3, 0 \rangle \cdot \langle 3, -2, 1 \rangle = 0$

$\overbrace{\langle 1, 1, -1 \rangle}^{\vec{v}} \cdot \langle 3, -2, 1 \rangle = 0$

$\underbrace{\langle 0, 10, 20 \rangle}_{\vec{v}} \cdot \langle 3, -2, 1 \rangle = 0$

e.g. $\vec{v} = \langle 2, 3, 0 \rangle$

ex Find a plane perpendicular to $3x - 2y + z = 0$ that passes through $(2, 0, 1)$.

$2x + 3y = 4$] $\vec{n} = \langle 2, 3, 0 \rangle$, $P_0 = (2, 0, 1)$

$\vec{n} = \langle 1, 1, -1 \rangle$, $P_0 = (2, 0, 1)$

$x + y - z = 1$] another plane

ex) Describe all vectors of the form $\langle 1, y, z \rangle$ that are perpendicular to $\langle 1, 1, -1 \rangle$.

$$\langle 1, 1, -1 \rangle \cdot \langle 1, y, z \rangle = 0$$

$$1 + y - z = 0$$

$$y - z = -1$$

$$y = z - 1$$

$$\boxed{\langle 1, z-1, z \rangle \text{ for any } z}$$

e.g.:

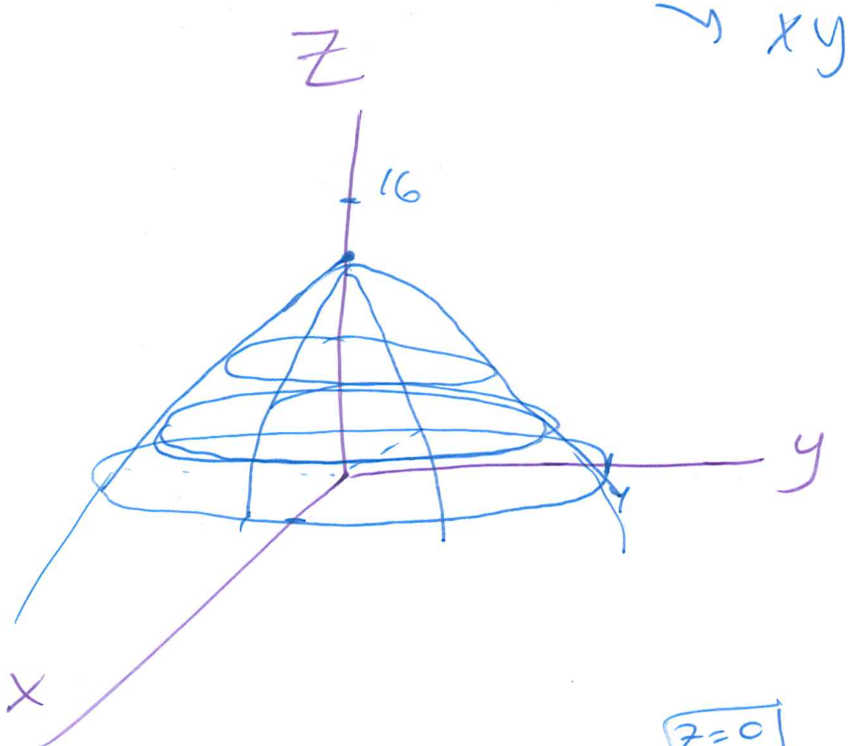
$\langle 1, -1, 0 \rangle$

$\langle 1, 0, 1 \rangle$

$\langle 1, 2, 3 \rangle$

Drawing Surfaces in $\underbrace{\mathbb{R}^3}$ using "traces"

$\to$ xyz system



ex $z = 16 - 4x^2 - y^2$

$0 \le \underbrace{4x^2 + y^2} = 16 - z$

$0 \le 16 - z$

$z \le 16$

$\boxed{z=0}$ $0 = 16 - 4x^2 - y^2 \to 4x^2 + y^2 = 16$

$\boxed{z=1}$ $1 = 16 - 4x^2 - y^2 \to 4x^2 + y^2 = 15$

$\boxed{z=2}$ $2 = 16 - 4x^2 - y^2 \to 4x^2 + y^2 = 14$

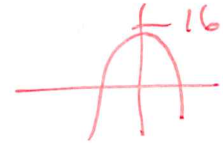
$\boxed{z=-10}$ $-10 = 16 - 4x^2 - y^2 \to 4x^2 - y^2 = 26$

what if x constant?

$$z = 16 - 4x^2 - y^2$$

$\boxed{x=0}$ $\quad \boxed{z = 16 - y^2}$

Think: $y = 16 - x^2$



$\boxed{x=1}$ $\quad z = 16 - 4 - y^2$

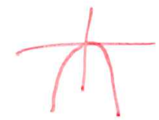
$\boxed{z = 12 - y^2}$

think: $y = 12 - x^2$

$\boxed{x=2}$ $\quad z = 16 - 4(4) - y^2$

$\boxed{z = -y^2}$

think: $y = -x^2$



$\boxed{\text{SHAPES}}$

$x^2 + y^2 = c$ circle

$ax^2 + by^2 = c$ ellipse

$ax^2 + by = c$ parabola

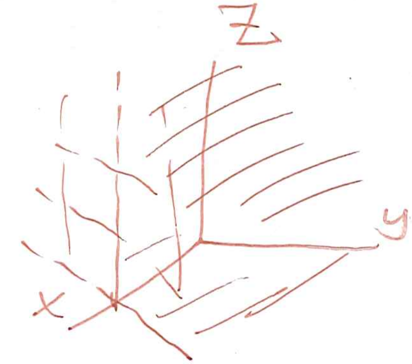
$ax + by = c$ line

A <u>level curve</u> of a surface is the intersection of the surface with a plane parallel to thy xy-plane.

We find these by setting z equal to a constant

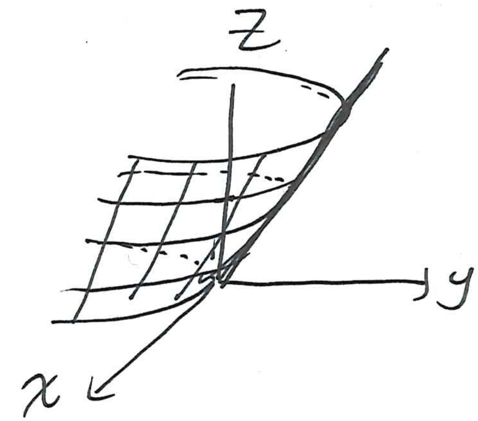
A <u>trace</u> of a surface is the intersection of the surface with a plane that is parallel to one of the coordinate planes.

$\boxed{z \to \text{constant}}$	xy-trace	(level curve)
$\boxed{y \to \text{constant}}$	xz-trace	
$\boxed{x \to \text{constant}}$	yz-trace	



Last time: $z = 16 - 4x^2 - y^2$

ex Graph $z = x^2 + y$



Level Curves ($z = $ constant)

If $z=0$: $0 = x^2 + y \rightarrow y = -x^2$ (parabola)

If $z=2$: $2 = x^2 + y \rightarrow y = 2 - x^2$

If $z=4$: $4 = x^2 + y \rightarrow y = 4 - x^2$

yz-trace

If $x=0$: $z = y$

Equations To Know in $\mathbb{R}^2$ (xy-plane)

$x^2 + y^2 = c$ circle

$ax^2 + by^2 = c$ ellipse

$ax^2 + by = c$ parabola

$ax + by = c$ line

Another one (not in syllabus)

$x^2 - y^2 = c$: hyperbola

) (

p 869: lots of examples (ignore last one)

A function $\overset{\text{dependent}}{z} = f(x,y)$ assigns a real value (output) to each input $\underbrace{(x,y)}_{\text{independent variables}}$ from some domain.

ex: $z = x + y$ only one z-value for a pair (x,y)

function

ex: $z^2 = x + y$

NOT A FUNCTION

ex: $\left.\begin{array}{l} x = 1 \\ y = 0 \end{array}\right\} \; x + y = 1 \; \left\} \begin{array}{l} z = 1 \\ z = -1 \end{array}\right.$

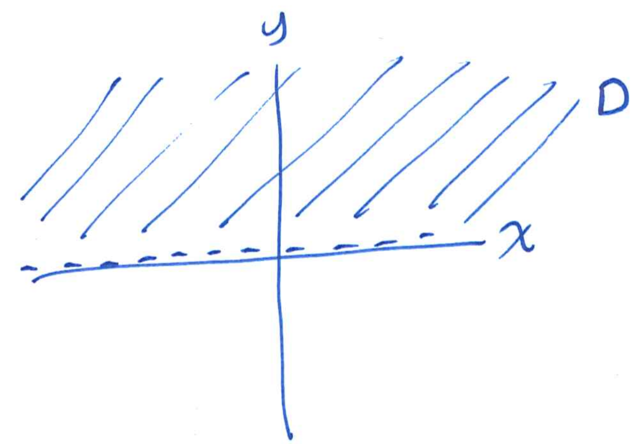
(ex) $f(x,y) = \sin\left(\frac{x}{\sqrt{y}}\right)$

DOMAIN:

Since we have $\sqrt{y}$: $y \geq 0$

Since $\frac{1}{\sqrt{y}} \leftarrow \sqrt{y} \neq y \quad y \neq 0$

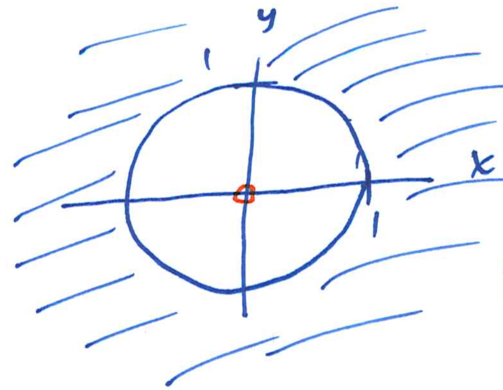
$\begin{cases} y > 0 \\ x \text{ any real number} \end{cases}$



RANGE: $[-1, 1]$

Usually, range of sine is $[-1, 1]$

$f(x,y) = \sqrt{y^2+x^2-1}$



DOMAIN

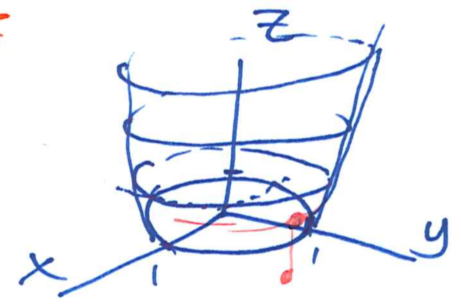
DOMAIN: $y^2+x^2-1 \geq 0$

$y^2+x^2 \geq 1$

example: $(0,0)$

$x=0 \quad y=0$

$\sqrt{0^2+0^2-1} = \sqrt{-1}$ DNE



RANGE: $[0, \infty)$

GRAPH: $z = \sqrt{y^2+x^2-1}$

Level Curves:

If $z=0$: $0 = \sqrt{y^2+x^2-1}$

$0 = y^2+x^2-1$

$1 = y^2+x^2$

If $z=1$: $1 = \sqrt{y^2+x^2-1}$

$1 = y^2+x^2-1$

$2 = y^2+x^2$

$z=2$: $2 = \sqrt{y^2+x^2-1}$

$4 = y^2+x^2-1$

$5 = y^2+x^2$

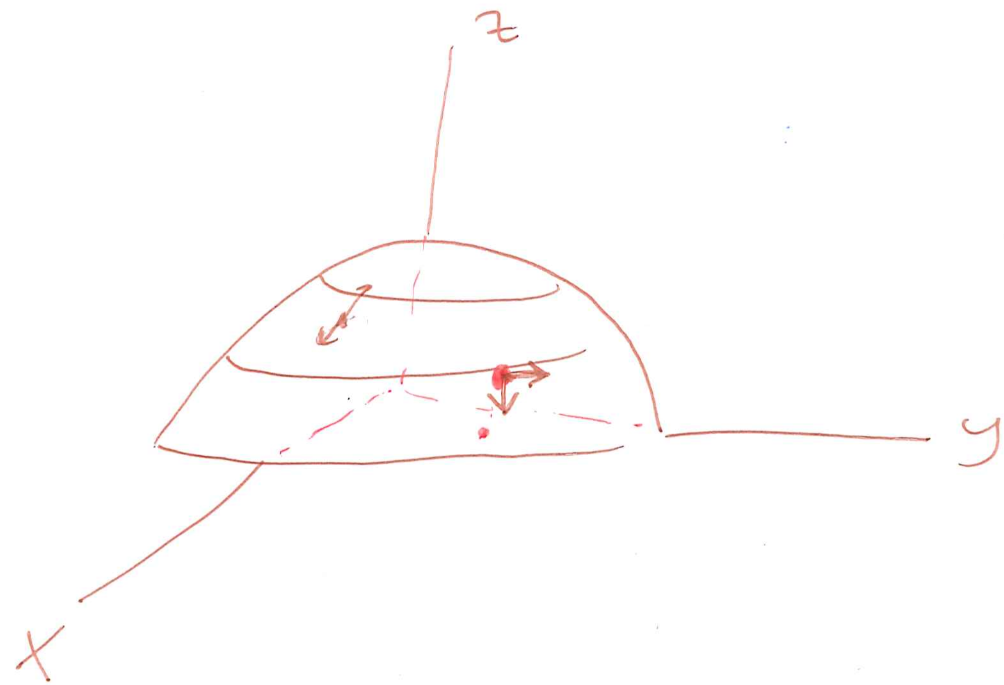
Recall:

$f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\frac{\text{change in output}}{\text{change in input}}$$



If we change x, but keep y same:

$$\frac{\text{change output}}{\text{change input}} : \frac{f(x+h, y)-f(x,y)}{h}$$

$$\boxed{\underbrace{f_x(x,y)}_{\text{partial derivative of } f \text{ with respect to } x} = \lim_{h \to 0} \frac{f(x+h, y)-f(x,y)}{h}}$$

Similarly, if we change y (not x):

$$\underline{\underline{f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h)-f(x,y)}{h}}}$$

Evaluate Partial Derivatives:

$$f(x,y) = 3xy^2 - 15x + \ln y$$

$f_x(x,y)$: treat y like constant

(also write: $\frac{\partial f}{\partial x}$)

$$f(x,y) = (3y^2)x - 15x + \ln y$$

$$f_x(x,y) = 3y^2 - 15$$

$$f(x,y) = (3x)y^2 - 15x + \ln y$$

$$f_y(x,y) = (3x) \cdot 2y + \frac{1}{y}$$

$$= 6xy + 1/y$$

(also write: $\frac{\partial f}{\partial y}$)

---

ex $f(x,y) = 2xy^2 - (5x+3y)^3$

$$f_x(x,y) = 2y^2 - 3(5x+3y)^2 \cdot 5$$

$$= \boxed{2y^2 - 15(5x+3y)^2}$$

economic interpretation of partial derivative :

$$P(K, L)$$

production (# items)

capital investment ($)

Labour (worker-hours)

P_K : $\lim_{h \to 0} \dfrac{P(K+h, L) - P(K, L)}{h}$ $\}\ \Delta P$ $\}\ \Delta K$

partial derivative with respect to K

$$\approx \frac{P(K+1, L) - P(K, L)}{1} = P(K+1, L) - P(K, L)$$

Change in productivity per dollar invested

P_L Change in productivity per hour worked

$P_L : \dfrac{\Delta P}{\Delta L}$ If $\Delta L = 1$, $P_L = \Delta P$

(ex) $G(W, C)$

cost of garment \$

cost of wool \$/lb

cost of cotton \$/lb

$G_W = 2$

$G_C = 5$

Question: If wool increases by \$3 / lb, then G increases by \$6 ?

$$\underline{2} = G_W \approx \frac{\Delta G}{\Delta w} = \frac{\Delta G}{\underline{3}} \implies \Delta G = 6$$

Higher-Order Derivatives

We always have to specify our "variable"

(ex) $f(x,y) = x\sin y$

1st Partial Derivs: $f_x = \sin y$ $\qquad$ $f_y = x\cos y$

2nd " " : $\boxed{f_{xy} = \cos y}$ $\qquad$ $f_{xx} = 0$ $\qquad$ $f_{yy} = -\sin y \cdot x$ $\qquad$ $\boxed{f_{yx} = \cos y}$

think: $(f_x)_y$

"mixed partials" (both variables)

Clairaut's Thm

Assume that f is defined on an open set D of $\mathbb{R}^2$ (x,y), and that f_{xy} and f_{yx} are continuous throughout D. Then:

$$f_{xy} = f_{yx}$$

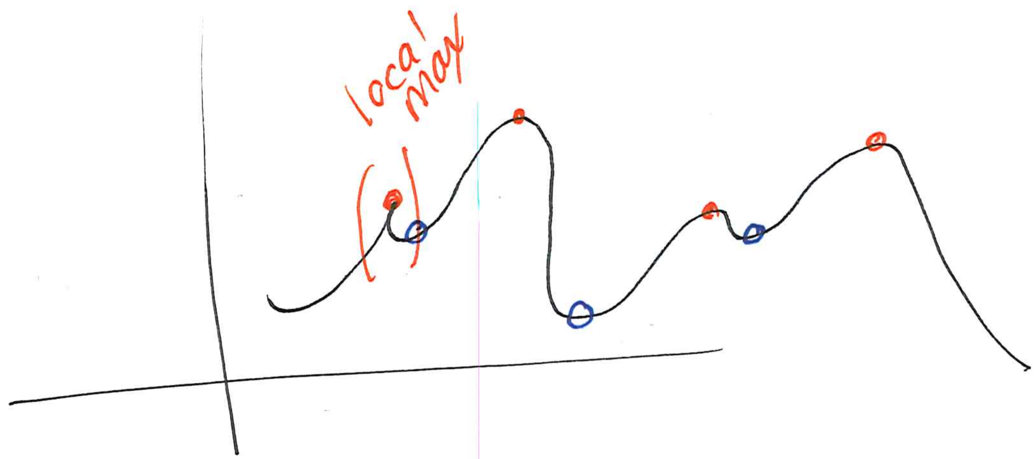
ex) Is it possible to have a function $f(x,y)$, defined everywhere, with $f_x = 3x$ and $f_y = 3x$?

Then: $f_{xy} = 0$ and $f_{yx} = 3$

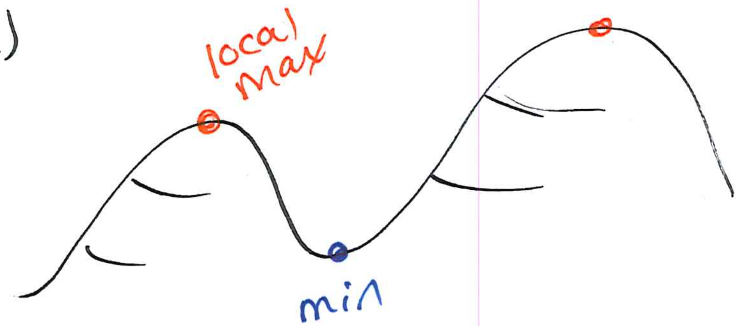
NOT POSSIBLE!

Ch. 12.8 Local Extrema (Max/Min Problems)

Recall:



$\mathbb{R}^3$ (xyz)



local max: can't walk uphill

local min: can't walk downhill

Thm) If f has a local max or min at (a,b), then if f_x and f_y exist at (a,b), then $f_x(a,b) = f_y(a,b) = 0$.

Def: If $f_x(a,b) = f_y(a,b) = 0$
OR if one DNE,
then (a,b) is a critical point.

ex) If $f_x = \frac{1}{x}$ and $f_y = 0$
then $(0,0)$ is a CP

ex) $z = 16 - 4x^2 - y^2$

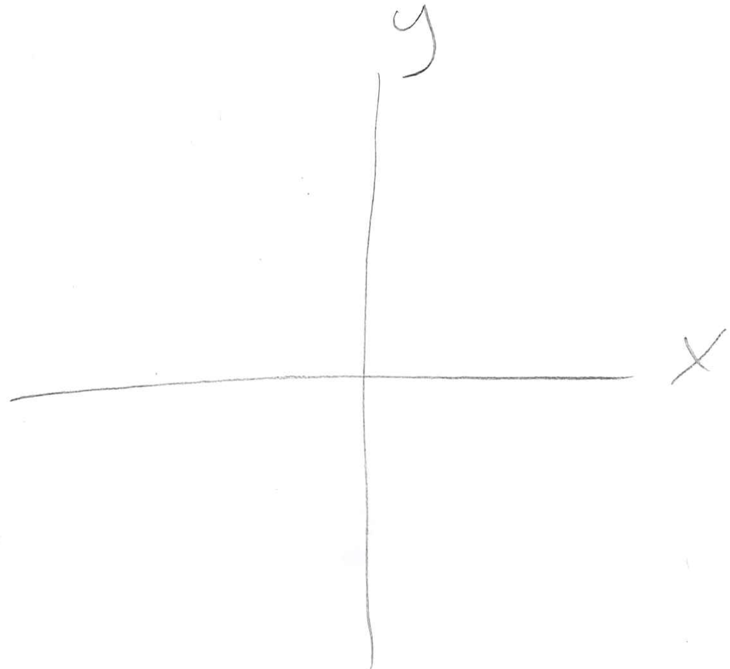
$\quad \| $

$f(x,y)$

$$\begin{cases} f_x = -8x \\ f_y = -2y \end{cases}$$

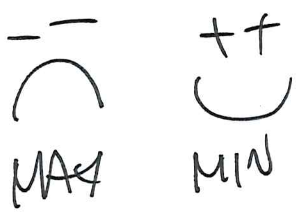
CP: $\begin{cases} 0 = -8x \\ 0 = -2y \end{cases}$

$$\boxed{CP: (0,0)}$$



Saddle pt: a point (a,b)
that is a critical pt,
but <u>not</u> a max or min

i.e. for any small neighborhood around (a,b):
there exists (x,y) such that



$$f(x,y) > f(a,b)$$ ← (a,b) not max

and there exists (x,y) such that
$$f(x,y) < f(a,b)$$ ← (a,b) not min

<u>Second Derivative Test</u>

Suppose the second partial derivatives of f are continuous near (a,b), and
$$f_x(a,b) = f_y(a,b) = 0$$

Define $D = f_{xx}(x,y) \cdot f_{yy}(x,y) - [f_{xy}(x,y)]^2$

① If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has local MAX at (a,b)

② If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then f has local MIN at (a,b)

③ If $D(a,b) < 0$, then (a,b) is a saddle point

④ If $D(a,b) = 0$, test inconclusive

(ex) $f(x,y) = x^2 - 2x - y^2 - 4y - 4$

Find & classify all CPs.

$$\begin{cases} f_x = 2x - 2 \\ f_y = -2y - 4 \end{cases}$$

$$\begin{cases} 0 = 2x - 2 \\ 0 = -2y - 4 \end{cases}$$

$$\begin{cases} x = 1 \\ y = -2 \end{cases}$$

Only CP: $(1,-2)$

Use 2nd Deriv Test

$f_{xx} = 2$

$f_{xy} = 0$

$f_{yy} = -2$

$$D = f_{xx} f_{yy} - f_{xy}^2$$
$$= (2)(-2) - 0$$
$$= -4 < 0$$

So: $(1,-2)$ is a saddle point

(ex) $f(x,y) = x^2 - 6x + \frac{1}{4}y^4 - 8y$

Find + classify CPs

$$\begin{cases} f_x = 2x-6 \\ f_y = y^3-8 \end{cases} \qquad \begin{cases} 0 = 2x-6 \\ 0 = y^3-8 \end{cases} \qquad \begin{matrix} x=3 \\ y=2 \end{matrix} \qquad \boxed{CP: (3,2)}$$

$D: f_{xx} f_{yy} - f_{xy}^2$

$f_{xx} = 2$

$f_{xy} = 0$

$f_{yy} = 3y^2$

$D(3,2) =$

$(2)(3 \cdot 2^2) - 0$

$= 2 \cdot 12 = 24 > 0$

$\boxed{D(a,b) > 0}$

$\boxed{f_{xx} > 0}$

$\boxed{(3,2) \text{ local min}}$

$++$

ex $f(x,y) = x^2 + xy^2 - 2x + 1$

Find ~~& classify~~ all CPs

$$\begin{cases} f_x = 2x + y^2 - 2 \\ f_y = 2xy \end{cases}$$

$$\begin{cases} 0 = 2x + y^2 - 2 \\ 0 = 2xy \end{cases}$$

$$\begin{cases} 0 = y^2 - 2 \\ x = 0 \end{cases} \quad \text{or} \quad \begin{cases} 0 = 2x - 2 \\ y = 0 \end{cases}$$

$$\begin{cases} y = \pm\sqrt{2} \\ x = 0 \end{cases} \quad \text{or} \quad \begin{cases} x = 1 \\ y = 0 \end{cases}$$

CP: $(0, \sqrt{2})$

$(0, -\sqrt{2})$

$(1, 0)$

$f_{xx} = 2$

$f_{xy} = 2y$

$f_{yy} = 2x$

$$D(x,y) = 2(2x) - (2y)^2$$
$$= 4x - 4y^2$$

$\boxed{(0, \sqrt{2})}$ $D(0,\sqrt{2}) = 0 - 4(2) < 0$ SADDLE PT

$\boxed{(0, -\sqrt{2})}$ $D(0,-\sqrt{2}) = 0 - 4(2) < 0$ SADDLE PT

$\boxed{(1,0)}$ $D(1,0) = 4 - 0 > 0$ LOCAL MIN

$f_{xx} = 2 > 0$

Using Inspection to Classify CPs

ex $f(x,y) = \cos(xy)$

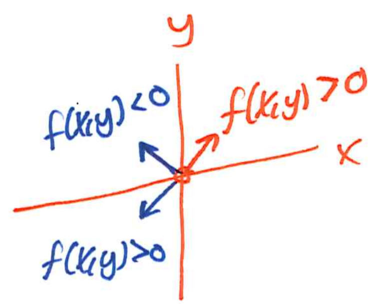
CP : $(0,0)$

$f(0,0) = 1$ MAX

(biggest cosine ever gets)

ex $f(x,y) = x \sin y$

CP : $(0,0)$



$(0,0)$ is a saddle pt

Inserted Section:

Partial Derivatives

using

Implicit Differentiation

z: "function"

$z = f(x, y)$

$$\underbrace{z \sin x} + \underbrace{y \sin z} = 0$$

differentiate with respect to x:

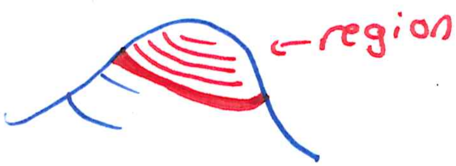
y - constant

z - function of x

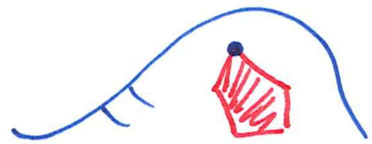
$$\underbrace{z \cdot \cos x + \sin x \cdot z_x} + \underbrace{y \cdot \cos z \cdot z_x} = 0$$

Absolute Maxima & Minima

(over a bounded region)



Max @ critical pt
(this example)



MAX @ boundary
(this example)

like endpoints

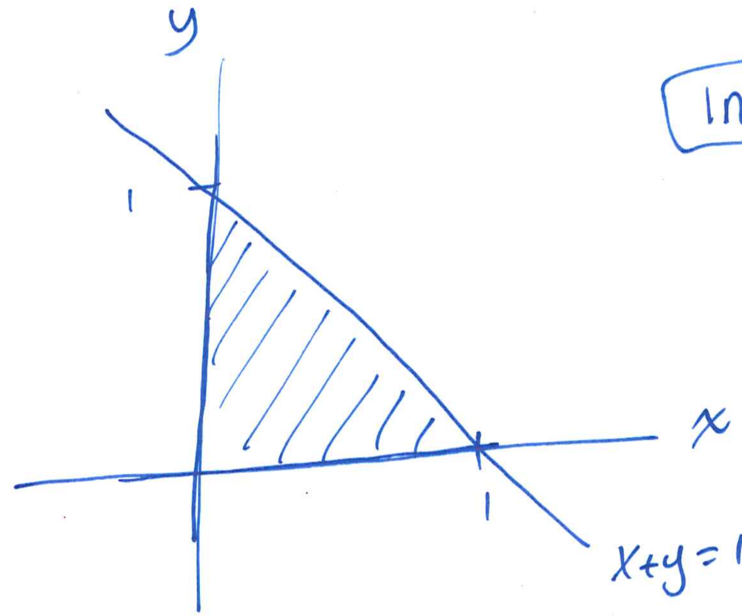
(ex) $f(x,y) = \dfrac{x+1}{x+y+1}$

Find absolute max + min values of $f(x,y)$

when:

- $x+y \le 1$
- $x \geq 0$
- $y \geq 0$

Inputs



Plan:

- Find CPs ← don't need 2nd deriv. test
- Find extrema along boundary
- Compare

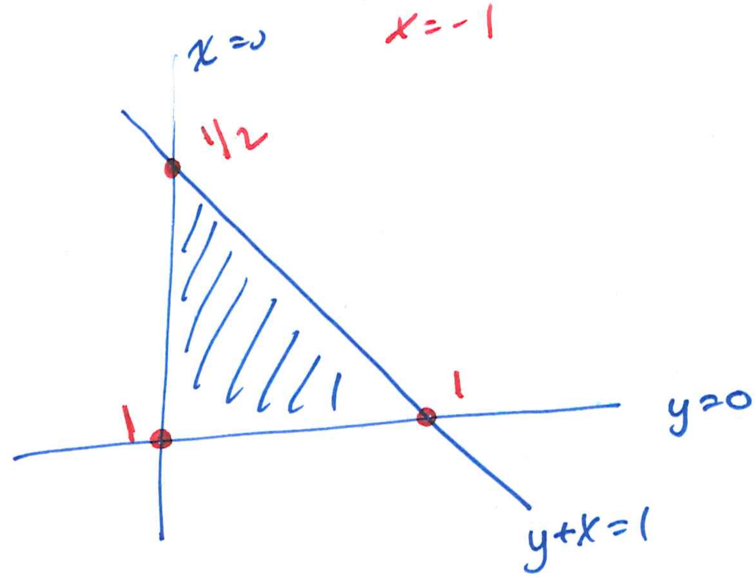
$\boxed{\text{Find CPs}}$ $\quad f(x,y) = \dfrac{x+1}{x+y+1}$

$$f_x = \frac{(x+y+1)(1) - (x+1)(1)}{(x+y+1)^2} = \frac{y}{(x+y+1)^2} = 0 \longrightarrow y = 0$$

$$f_y = \frac{(x+y+1)(0) - (x+1)(1)}{(x+y+1)^2} = \frac{-(x+1)}{(x+y+1)^2} = 0 \longrightarrow x = -1$$

$-(x+1) = 0$
$x+1 = 0$
$x = -1$

CP: $\left.\begin{array}{l} x=-1 \\ y=0 \end{array}\right\}$ not in region — ignore



$\boxed{\text{Check Boundary}}$

$\boxed{y+x=1}$ $\quad 0 \le x \le 1$

In that case:

$$f(x,y) = \frac{x+1}{x+y+1} = \frac{x+1}{2}$$

$$f(0,1) = \frac{0+1}{2} = 1/2$$

$$f(1,0) = \frac{1+1}{2} = 1$$

$\boxed{x=0}$ $\quad 0 \le y \le 1$

$$f(0,y) = \frac{0+1}{0+y+1} = \frac{1}{y+1}$$

$$f(0,1) = \frac{1}{1+1} = \frac{1}{2}$$

$$f(0,0) = \frac{1}{0+1} = 1$$

$\boxed{y=0}$ $\quad 0 \le x \le 1$

$$f(x,0) = \frac{x+1}{x+0+1} = \frac{x+1}{x+1} = 1$$

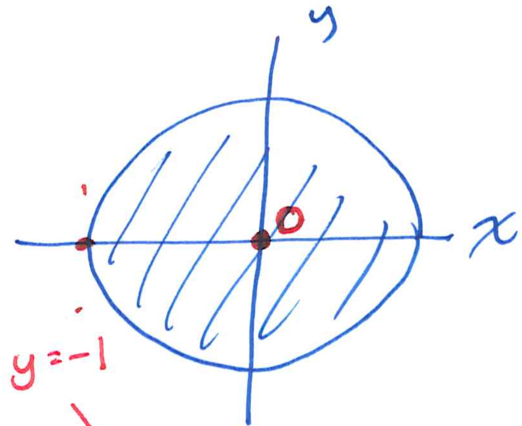
MAX: $f(x,y) = 1$ when $y = 0$

MIN: $f(x,y) = 1/2$ at $(0,1)$

ex $f(x,y) = x^2 + x^2y + y^2$

Find absolute extrema when $x^2+y^2 \le 1$

max/min



Critical Points

$$\begin{cases} f_x = 2x + 2xy = 2x(1+y) = 0 \longrightarrow x=0 \text{ OR } y=-1 \\ f_y = x^2 + 2y = 0 \longrightarrow \end{cases}$$

If $x=0$:
$2y=0$
$y=0$

If $y=-1$:
$x^2-2=0$
$x^2=2$
$x=\pm\sqrt{2}$

$f(0,0)=0$

CP: $(0,0)$
$\cancel{(-\sqrt{2},-1)}$
$\cancel{(\sqrt{2},-1)}$ not in region

$\boxed{\text{Boundary}}$ If (x,y) is a point of boundary (not inside):

$x^2+y^2=1 \longrightarrow x^2=1-y^2$

Then: $f(x,y) = x^2+x^2y+y^2$

$= (1-\cancel{y^2}) + (1-y^2)y + \cancel{y^2}$

$= 1+y-y^3 \leftarrow$ where is this MAX / MIN

$-1 \le y \le 1$

Subproblem: $g(y) = -y^3+y+1$

$-1 \le y \le 1$

Find extrema

$g'(y) = -3y^2+1 = 0$

$1 = 3y^2$

$\frac{1}{3} = y^2 \qquad y = \pm\frac{1}{\sqrt{3}}$

$g(1) = -1+1+1 = 1$

$g(-1) = -(-1)-1+1 = 1-1+1 = 1$

$g\left(\frac{1}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}^3} + \frac{1}{\sqrt{3}} + 1 = \frac{1}{\sqrt{3}}\left(1-\frac{1}{3}\right)+1 = \boxed{\frac{2}{3\sqrt{3}}+1}$ MAX: when $y = \frac{1}{\sqrt{3}}$

$g\left(\frac{-1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}^3} - \frac{1}{\sqrt{3}} + 1 = \frac{1}{\sqrt{3}}\left(\frac{1}{3}-1\right)+1 = \boxed{\frac{-2}{3\sqrt{3}}+1}$ MIN: when $y = \frac{-1}{\sqrt{3}}$

Compare:

interior $\{\ f(0,0) = 0$] ABSOLUTE MIN

boundary $\begin{cases} f(\ \ ,\frac{1}{\sqrt{3}}) = \frac{2}{3\sqrt{3}} + 1 \\ f(\ \ ,\frac{-1}{\sqrt{3}}) = 1 - \frac{2}{3\sqrt{3}} \end{cases}$] ABSOLUTE MAX (first line)

$y = \frac{1}{\sqrt{3}} \rightarrow y^2 = \frac{1}{3}$

$x^2 = 1 - y^2 = 1 - \frac{1}{3} = 2/3$

$\boxed{x = \pm\sqrt{\frac{2}{3}}}$

$\boxed{y = \frac{1}{\sqrt{3}}}$

Which is the smallest value?

$3\sqrt{3} > 3$

So: $\frac{1}{3\sqrt{3}} < \frac{1}{3}$

So: $\frac{2}{3\sqrt{3}} < \frac{2}{3}$

So: $\frac{-2}{3\sqrt{3}} > \frac{-2}{3}$

So: $1 - \frac{2}{3\sqrt{3}} > 1 - \frac{2}{3} = 1/3 > 0$

0 is the smallest value.

ex

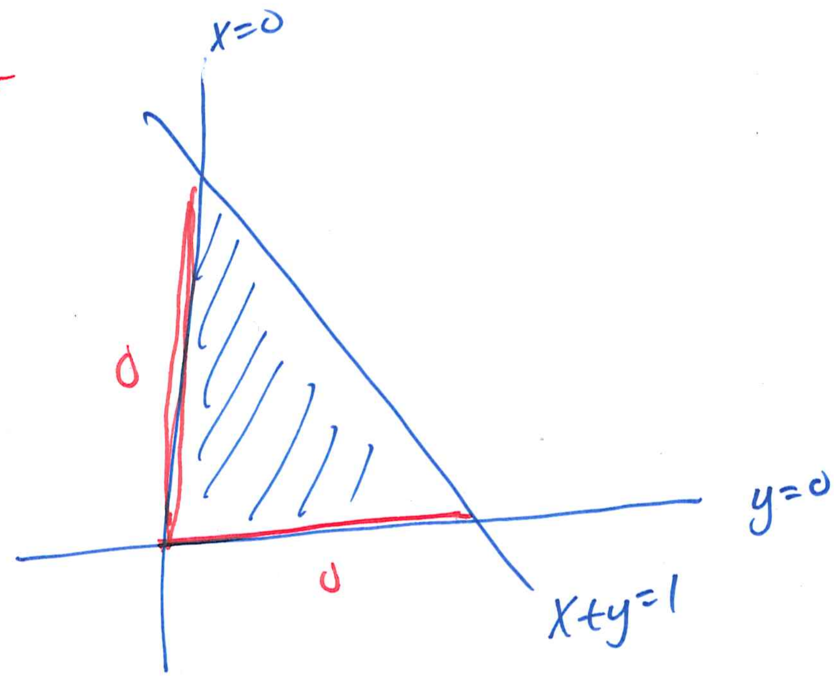
$f(x,y) = (xy)e^{-x-y}$

Region:

$R = \{ \underbrace{(x,y)}_{\text{of all of these}} : \underbrace{x \geq 0,\ y \geq 0,\ x+y \leq 1}_{\text{these true}} \}$

collection

such that



Find abs max/min of f(x,y) over R

Plan:

- Find CPs
- Find extrema on boundary
- Compare

$\boxed{\text{CPs}}$ $f(x,y) = (xy)e^{-x-y}$

$f_x = (xy)\cdot e^{-x-y}(-1) + e^{-x-y}\cdot y$

$= ye^{-x-y}(1-x) = 0 \longrightarrow \boxed{y=0 \quad \underline{\text{OR}} \quad x=1}$

AND

$f_y = xe^{-x-y}(1-y) = 0 \longrightarrow \boxed{x=0 \quad \underline{\text{OR}} \quad y=1}$

CP: $\cancel{(1,1)}$ not in R

$\boxed{(0,0)}$ $\boxed{f(0,0)=0}$

$$f(x,y) = (xy)e^{-x-y}$$

$\boxed{\text{Boundaries}}$

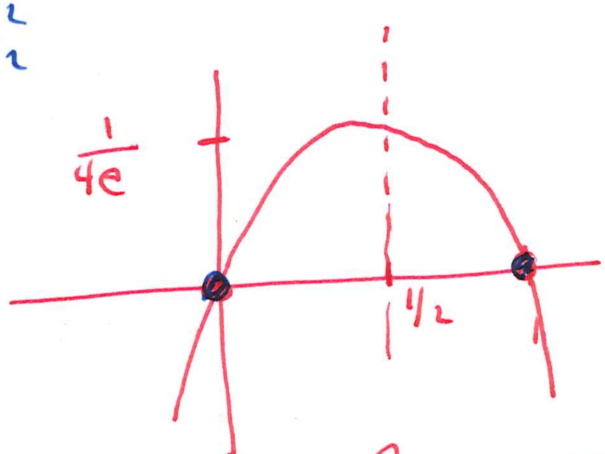
$\boxed{x=0}$ $\boxed{y=0}$ $f(x,y)=0$

$\boxed{x+y=1}$ $f(x,y) = (xy)e^{-(x+y)} = xye^{-1} = \frac{xy}{e} = \underbrace{\frac{1}{e}x(1-x)}_{\text{parabola } \cap}$

$y = 1-x \rightarrow$ if $x = 1/2$ then $y = 1/2$

$0 \le x \le 1$

intercepts: $x=0$, $x=1$



MIN: when $x=0$ or $x=1$, fcn $= 0$

MAX: when $\boxed{x = 1/2}$

fcn: $\frac{1}{e}\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right) = \frac{1}{e}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4e}$

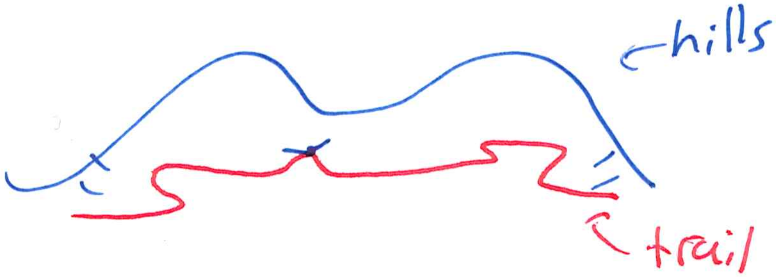
$\boxed{\text{COMPARE}}$

CP: $f(0,0) = 0$

Bdry: If $x=0$ or $y=0$, $f(x,y)=0$ $\Big\}$ 0 : MIN

If $x = 1/2$ $y = 1/2$, $f(1/2, 1/2) = \frac{1}{4e}$ $\Big\}$ MAX

Ch 12.9 Method of Lagrange Multipliers (Constrained optimization)



find highest pt of trail (boundary)

- "We can use this method for abs max/min over a bounded region when boundary is hard to plug in."

Method of Lagrange Multipliers

Suppose we want to find abs max/min of a function

$f(x,y)$ } "objective function"

subject to the constraint

$g(x,y) = \underset{\substack{\uparrow \\ \text{constant}}}{c}$

These will only occur at points (a,b) such that, for some constant λ, all three equations below are true:

$$\begin{cases} ① \quad f_x(a,b) = \lambda g_x(a,b) \\ ② \quad f_y(a,b) = \lambda g_y(a,b) \end{cases}$$

③ $g(x,y) = c$

How do we solve these 3 equations?

① $g_x(a,b)=0$ $\boxed{\text{OR}}$ $\lambda = \dfrac{f_x(a,b)}{g_x(a,b)}$

② $g_y(a,b)=0$ $\boxed{\text{OR}}$ $\lambda = \dfrac{f_y(a,b)}{g_y(a,b)}$

MAYBE: $g_x(a,b)=0$ and $f_x(a,b)=0$

MAYBE: $g_y(a,b)=0$ and $f_y(a,b)=0$

MAYBE: $\dfrac{f_x(a,b)}{g_x(a,b)} = \dfrac{f_y(a,b)}{g_y(a,b)}$

ALSO: $g(x,y)=c$

Quiz 2:

Tuesday, Jan 24

12.2 – 12.9

Similar to suggested problems
in book
(modified for time)

Quiz 1:
Hopefully in MLC
after 4 PM

Worksheet to practice optimization is on the course website

Method of Lagrange Multipliers

Let the objective function f and the constraint function $g(x,y) = c$ be differentiable, with
constant

$g_x(x,y)$ and $g_y(x,y)$ not both always zero.

To locate the maximum and minimum values of f subject to the constraint $g(x,y) = c$:

① Find the values of x, y, and λ that satisfy the equations:

$$\begin{cases} f_x(x,y) = \lambda\, g_x(x,y) \\ f_y(x,y) = \lambda\, g_y(x,y) \\ g(x,y) = c \end{cases}$$

② Among the values (x,y) from Step 1, select the largest & smallest corresponding function values. These are the maximum & minimum values of f subject to the constraint.

ex The height of a roller coaster at (x,y) is

$$f(x,y) = xy + 14$$

But it only exists when $x^2+y^2=18$

What are highest & lowest points?

Objective (want to maximize / minimize)

$$f(x,y) = xy + 14$$

Constraint: $g(x,y) = x^2+y^2 = 18$

$f_x = y$ $\quad$ $g_x = 2x$

$f_y = x$ $\quad$ $g_y = 2y$

Solve:

$$\begin{cases} y = \lambda \cdot \underbrace{2x} \\ x = \lambda \cdot \underbrace{2y} \\ x^2+y^2=18 \end{cases}$$

$\cancel{2x=0}$ OR $\lambda = \frac{y}{2x}$

$\cancel{2y=0}$ OR $\lambda = \frac{x}{2y}$

If $2x=0$: Then $x=0$

1st Equation: $y=0$

Point $(0,0)$ NOT in constraint

No point to consider.

If $2y=0$: Then $y=0$

2nd Equation: $x=0$

$(0,0)$ not in constraint

No point to consider

So: $\frac{y}{2x} = \lambda = \frac{x}{2y}$, so $2y^2 = 2x^2$, so $x^2=y^2$

3rd Equation: $x^2+x^2=18$

$2x^2=18$

$x^2=9$

$x=\pm 3$

Since $y^2=x^2$

$y^2=9$

$y=\pm 3$

Four points to consider: $(\pm 3, \pm 3)$

$f(x,y) = xy + 14$

$f(3,3) = f(-3,-3) = 9 + 14 = \boxed{23}$ highest pt (abs max)

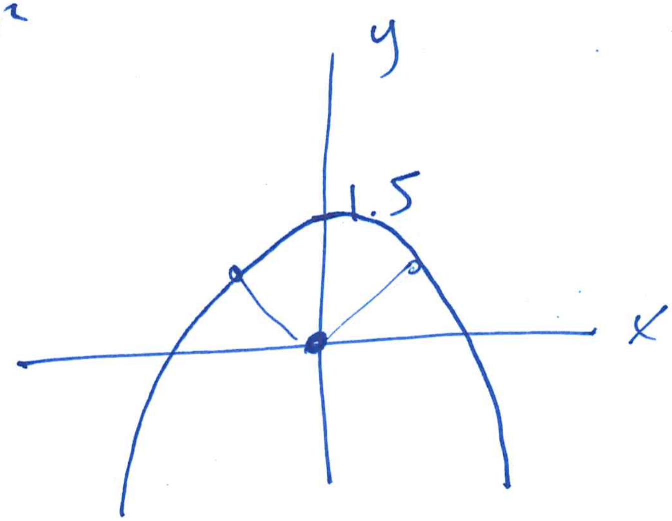
$f(3,-3) = f(-3,3) = -9 + 14 = \boxed{5}$ lowest pt (abs min)

along constraint

ex Find the point(s) on the parabola

$$y = 1.5 - x^2$$

closest to origin.



Want to minimize: distance to (0,0)

For any point (x,y):

$$D = \sqrt{x^2+y^2}$$

Easier:

$f(x,y) = x^2+y^2$ objective function

Only care about (x,y) if on parabola: $y = 1.5 - x^2$

$$y + x^2 = 1.5$$

Constraint:

$$g(x,y) = y + x^2 = 1.5$$

$f_x = 2x$ $\quad$ $g_x = 2x$

$f_y = 2y$ $\quad$ $g_y = 1$

Solve:

$$\begin{cases} 2x = \lambda \cdot \underbrace{2x} \\ 2y = \lambda \cdot 1 \\ y + x^2 = 1.5 \end{cases}$$

$\rightarrow$ $2x = 0$ $\quad$ OR $\quad$ $\lambda = \frac{2x}{2x} = 1$

$\rightarrow$ $\lambda = 2y$

If $2x = 0$: $(x=0)$

3^{rd} Equation: $y + 0 = 1.5$

$y = 1.5$

Point to Consider: $(0, 1.5)$

Otherwise: (1^{ST}) $\lambda = 1$ and (2^{nd}) $\lambda = 2y$

$1 = 2y$, so $y = 1/2$

3^{rd}: $\frac{1}{2} + x^2 = 1.5$

$x^2 = 1$

$x = \pm 1$

Points to consider: $(1, \frac{1}{2})$

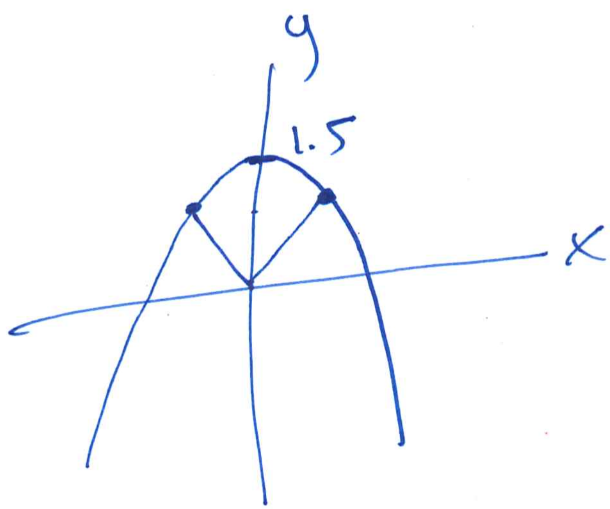
$(-1, \frac{1}{2})$

$f(x,y) = x^2+y^2$

- $f(0, 1.5) = 0^2 + \left(\frac{3}{2}\right)^2 = \frac{9}{4}$
- $f(1, \frac{1}{2}) = 1^2 + \left(\frac{1}{2}\right)^2 = 1 + \frac{1}{4} = \frac{5}{4}$
- $f(-1, \frac{1}{2}) = (-1)^2 + \left(\frac{1}{2}\right) = \frac{5}{4}$

$\left.\begin{array}{l} f(1,\frac{1}{2}) \\ f(-1,\frac{1}{2}) \end{array}\right\}$ smaller

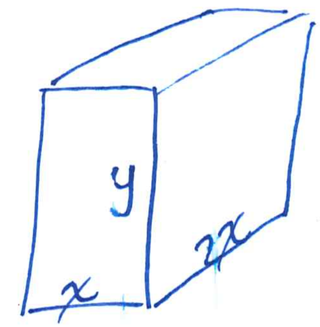
Closest points: $(1, \frac{1}{2})$ & $(-1, \frac{1}{2})$



ex A rectangular box has volume 72 cu ft.

Its width is twice its length.

What is the minimum possible surface area, and what dimensions achieve this?



Objective fcn: surface area

surface area:

$$2(xy)+2(2xy)+2(2x^2)$$
$$= 2xy+4xy+4x^2$$
$$= 6xy+4x^2$$

$$f(x,y) = 6xy+4x^2$$

$f_x = 6y+8x$ $\quad g_x = 4xy$

$f_y = 6x$ $\quad g_y = 2x^2$

Constraint: volume

$$g(x,y) = 2x^2y = 72$$

Solve:

$$\begin{cases} 6y+8x = \lambda \cdot \underbrace{4xy} & \rightarrow \ \cancel{4yx=0} \quad \text{OR} \quad \lambda = \dfrac{6y+8x}{4xy} = \dfrac{3y+4x}{2xy} \\ 6x = \lambda \cdot 2x^2 & \rightarrow \ \cancel{2x^2=0} \quad \text{OR} \quad \lambda = \dfrac{6x}{2x^2} = \dfrac{3}{x} \\ 2x^2y = 72 \end{cases}$$

By Eqn #3: $x \neq 0$, $y \neq 0$

So $4yx \neq 0$ and $2x^2 \neq 0$

So: $\overset{(1st)}{\dfrac{3y+4x}{2xy}} = \lambda = \overset{(2nd)}{\dfrac{3}{x}}$

$3xy + 4x^2 = 6xy$

$4x^2 = 3xy$ Since $x \neq 0$, cancel it

$4x = 3y$

$\cancel{2x = 3y}$

$\cancel{x = \frac{3}{2}y}$

$y = \frac{4}{3}x$

3rd Eqn: $2x^2\left(\frac{4}{3}x\right) = 72$

$\frac{\cancel{8}}{3}x^3 = 72 = 9 \cdot \cancel{8}$

$x^3 = 27$

$x = 3$ → $y = \frac{4}{3}(3)$

$y = 4$

Point: $(3,4)$

1 option: There is no max surface area, so (3,4) must give min

Another option: Choose any other point. (on constraint)

ex: $2x^2y = 72$

$x = 1$
$y = 36$ (one of many possible choices)

Surface area: $f(1,36) = 6(1)(36) + 4(1)^2$
$= 6 \cdot 36 + 4 = 18 \cdot 12 + 4$ ← bigger

Compare: $f(3,4) = 6(3)(4) + 4(3)^2$
$= 6 \cdot 12 + 12 \cdot 3$
$= 12 \cdot 9$ ← NOT MAX

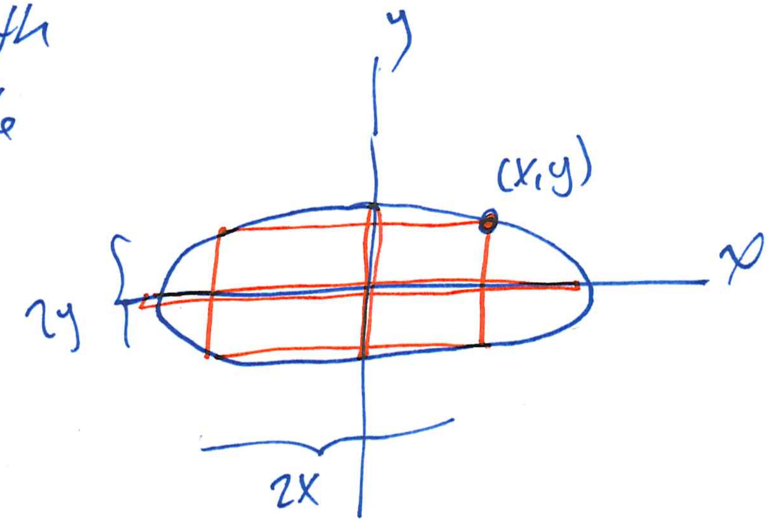
MIN surface area: $12 \cdot 9$ sq ft

Dimensions: $(3) \times (6) \times (4)$
$x \quad 2x \quad y$

Third option: Question asked for a min; this should be a min ex What is the maximum area of a rectangle, with sides parallel to coordinate axes, inscribed in the ellipse

$$4x^2+16y^2=16$$

?



Objective (maximize): Area

Area: $(2x)(2y)$ if (x,y) is the point in 1st quadrant where rect, ellipse touch

$f(x,y)=4xy$ $(4|x||y|)$
$(x,y>0)$

Constraint: $g(x,y)=4x^2+16y^2=16$

$f_x=4y$ $g_x=8x$

$f_y=4x$ $g_y=32y$

Solve:

$$\begin{cases} 4y = \lambda \cdot \underbrace{8x} \\ 4x = \lambda \cdot \underbrace{32y} \\ 4x^2+16y^2=16 \end{cases}$$

$4y = \lambda \cdot 8x \rightarrow \cancel{8x=0}$ or $\lambda = \frac{4y}{8x} = \frac{y}{2x}$

$4x = \lambda \cdot 32y \rightarrow \cancel{32y=0}$ or $\lambda = \frac{4x}{32y} = \frac{x}{8y}$

If $8x=0$, then $x=0$, 1^{ST} Eq: $y=0$ BUT: 3^{rd} Eq: not in constraint

If $32y=0$, then $y=0$, 2^{ND} Eq: $x=0$ BUT: $(0,0)$ not in constraint

Last case: $\underset{(1st)}{\frac{y}{2x}} = \lambda = \underset{(2nd)}{\frac{x}{8y}}$

$8y^2 = 2x^2$

$4y^2 = x^2$

$16y^2 = 4x^2$

3^{rd} Eq: $4x^2 + (4x^2) = 16$

$8x^2 = 16$

$x^2 = 2$

$x = \pm\sqrt{2}$

$4y^2 = 2$

$y^2 = \frac{1}{2}$

$y = \pm\frac{1}{\sqrt{2}}$

Points to Consider:

$\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}\right)$

$f(x,y) = 4xy$

$f(\sqrt{2}, \frac{1}{\sqrt{2}}) = 4(\sqrt{2})(\frac{1}{\sqrt{2}}) = 4$

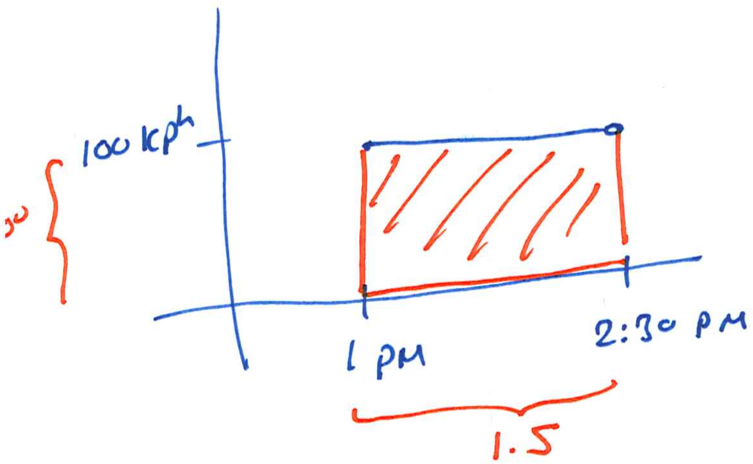
$\cancel{f(-\sqrt{2}, \frac{-1}{\sqrt{2}}) = 4}$

we chose x,y in 1st quadrant so x,y ≥ 0

4: max area.

Ch. 5.1 : Approximating Area under Curves

DISTANCE = RATE × TIME

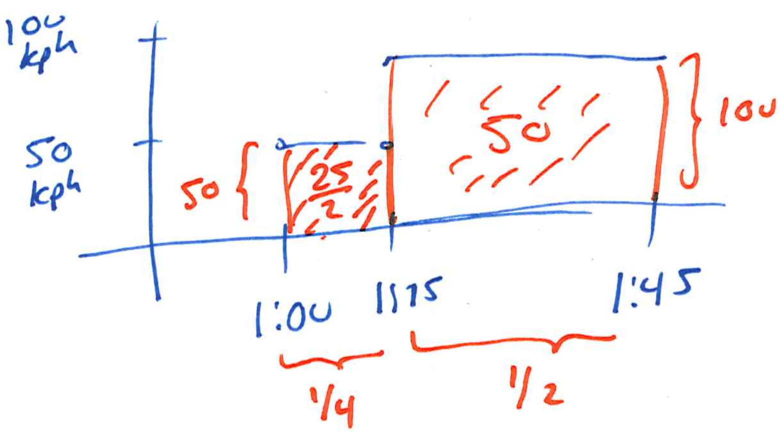


Dist travelled from 1 pm to 2:30 pm:

$(100 \frac{km}{hr})(1.5 \text{ hr})$

$= \boxed{150 \text{ km}}$

Area of rectangle under line



Dist travelled from 1 – 1:45 :

1–1:15 : $\frac{1}{4}$ hr, 50 kph

dist : $\frac{50}{4} = \frac{25}{2}$ km

1:15–1:45 : $\frac{1}{2}$ hr, 100 kph

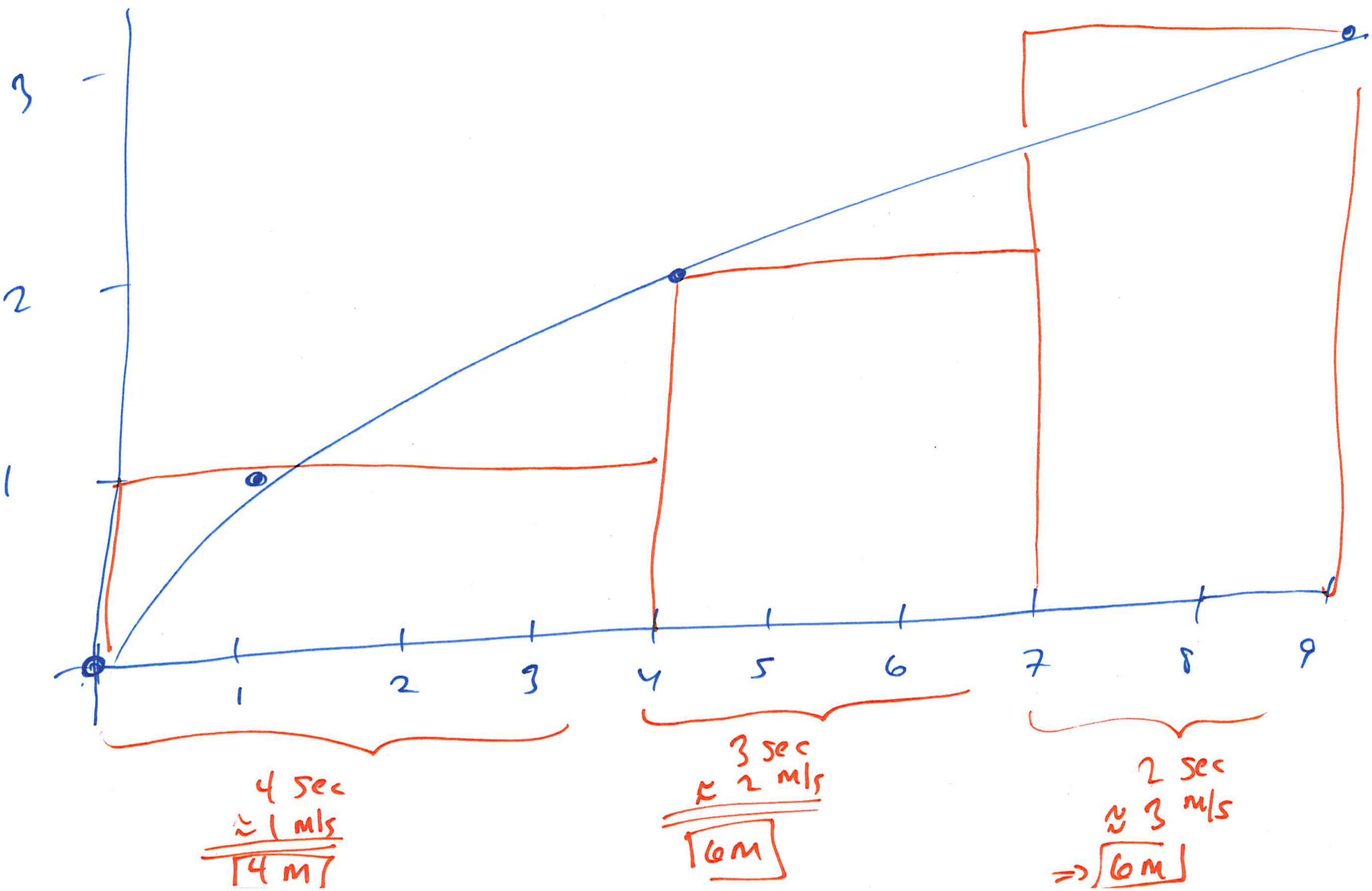
dist : $(\frac{1}{2})(100) = 50$ km

All together : $\boxed{50 + \frac{25}{2} \text{ km}}$

DISTANCE TRAVELLED: AREA UNDER LINE

At time t,
speed:
$S(t) = \sqrt{t}$ m/s

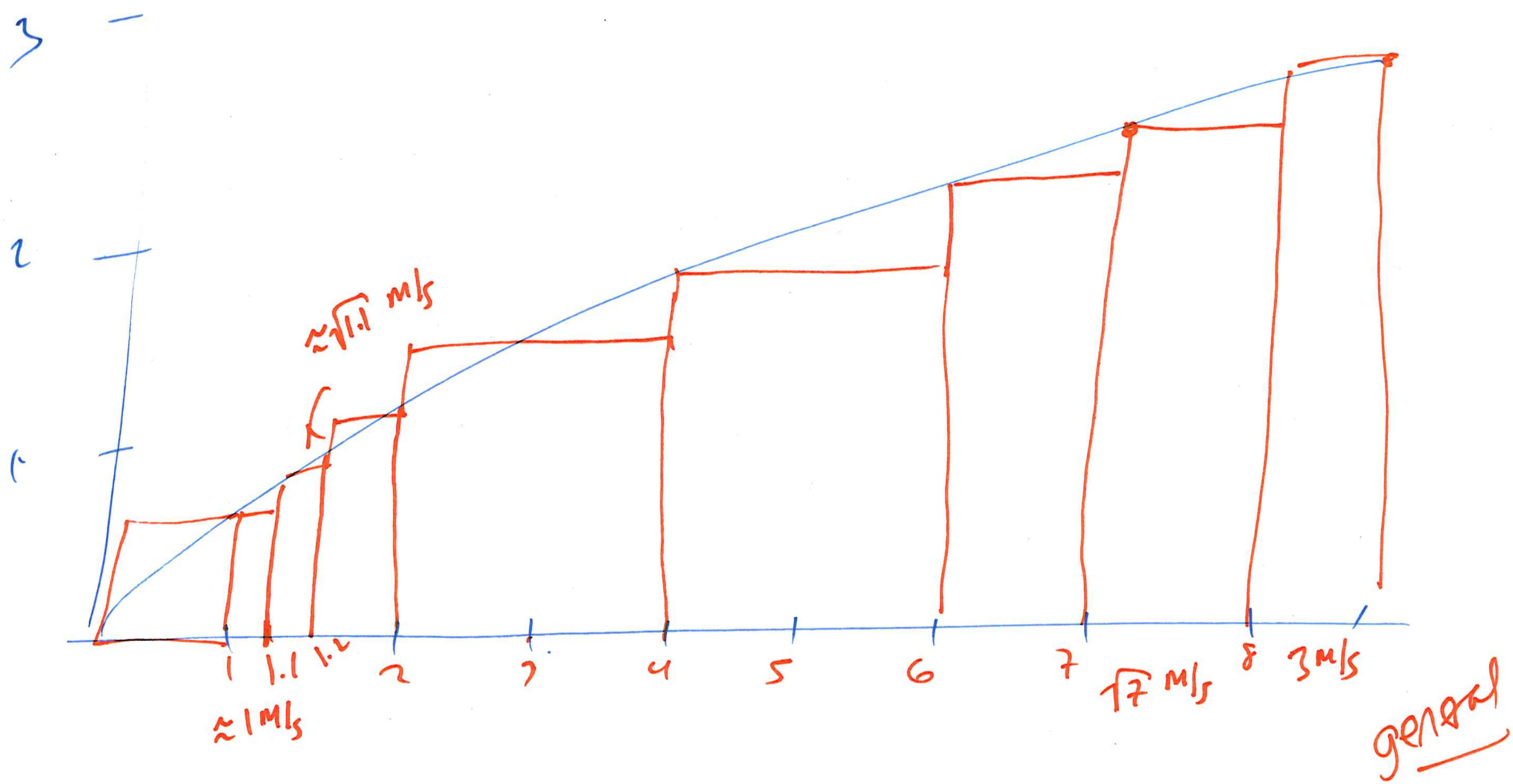
APPROX DIST TRAVELLED
≈ 16 m



Grid pointss 0, 1, 1.1, 1.2, 2, 4, 6, 7, 8, 9

Partition: Regular if cuts all same size
width of intervals: Δx

General if not same size

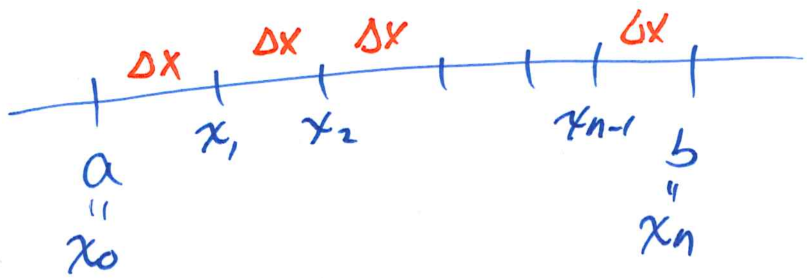


In a regular partition from a to b, with n intervals:

$$\Delta x = \frac{b-a}{n}$$

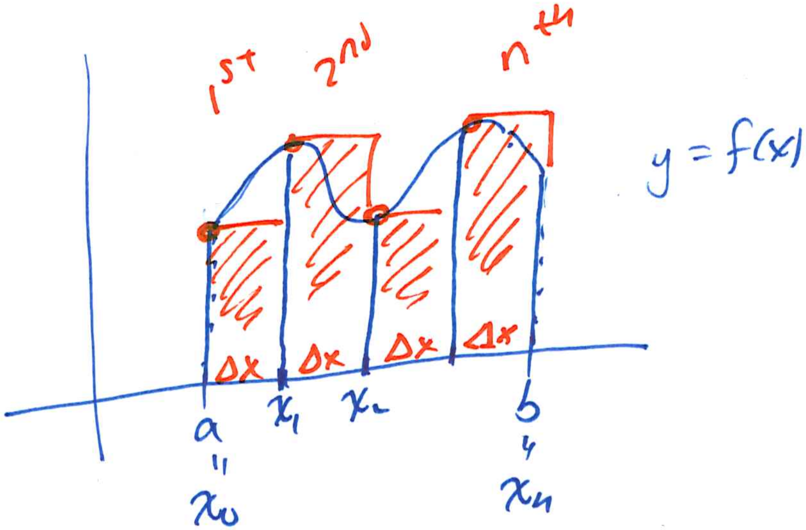
Grid points:

$$x_i = a + i\,\Delta x$$



Riemam Sums

Regular partitions



Left Riemann Sum

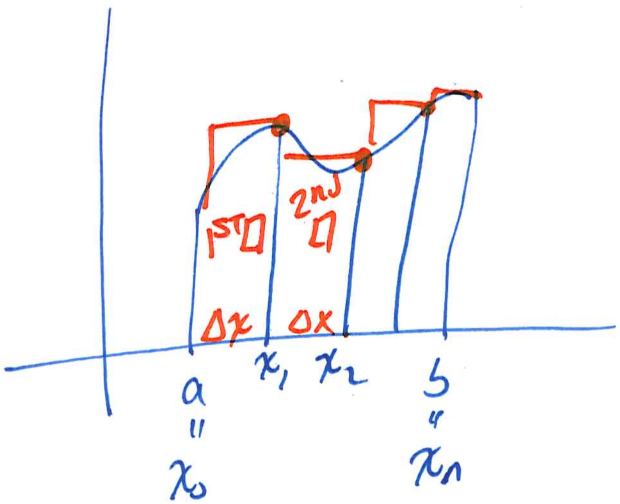
Area of i^{th} rectangle:

$= (\text{base})(\text{height})$

$= \Delta x \cdot f(x_{i-1})$

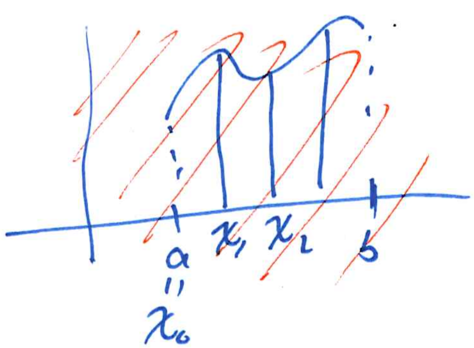
Approx of tl area:

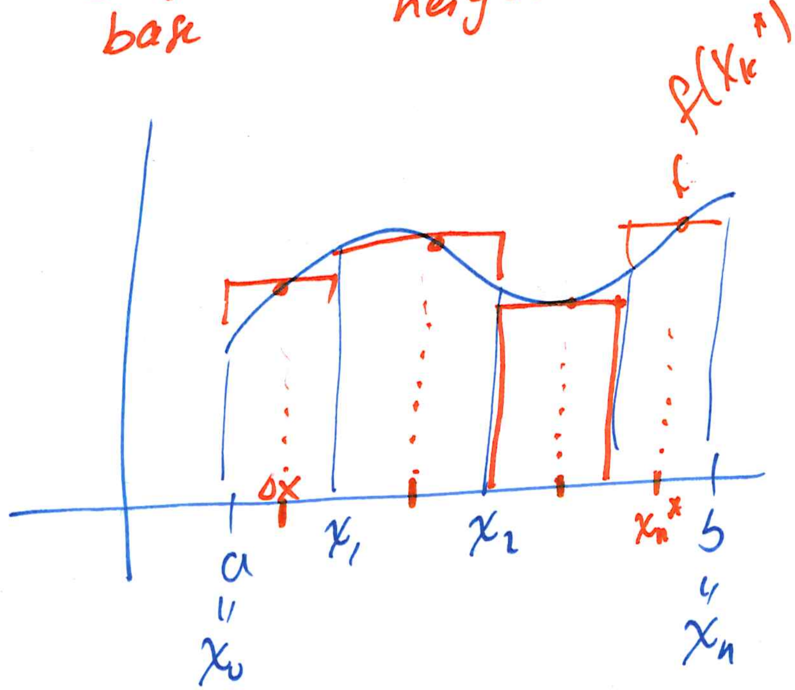
$\Delta x f(x_0) + \Delta x\, f(x_1) + \Delta x\, f(x_2) + \cdots + \Delta x\, f(x_{n-1})$



Right Rieman Sum

Area of i^{th} □ : $\underbrace{(\Delta x)}_{\text{base}} \cdot (\underbrace{f(x_i)}_{\text{height}})$





Midpoint Riemann Sum

height of i^{th} rectangle:

$$f\left(\frac{x_{i-1}+x_i}{2}\right)$$

base: ΔX

Riemann Sum

Suppose f is defined on a closed interval $[a,b]$, which is divided into n intervals of equal length, Δx.

If x_k^* is any point in the k^{th} interval $[x_{k-1}, x_k]$, for $k=1,\ldots,n$, then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

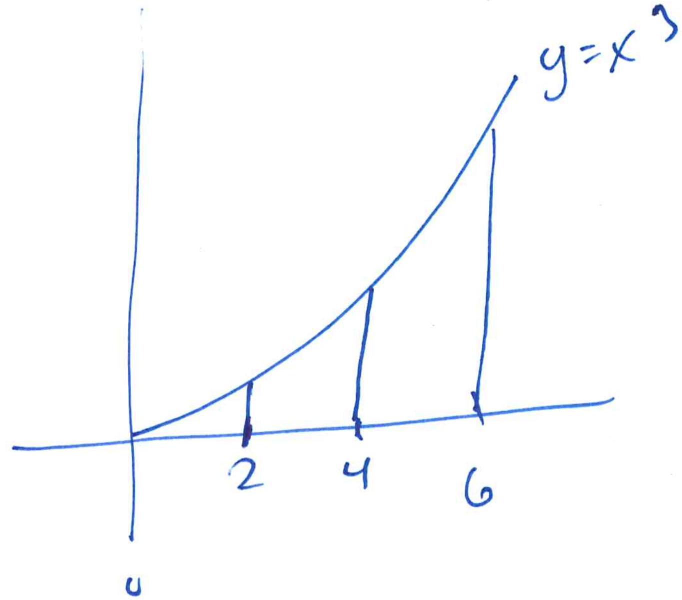
is called a <u>Riemann Sum</u> of f on $[a,b]$.

It's called a: $\left(\begin{array}{c}\text{left}\\ \text{right}\\ \text{midpoint}\end{array}\right)$ Riemann sum if

x_i^* is the (") of the interval $[x_{i-1}, x_i]$.

(ex) $y = x^3$

Approx area under curve, on $[0,6]$, using 3 subintervals & Riemann Sum.



$n = 3$

$\Delta x = \frac{6-0}{3} = 2$

width of each partition

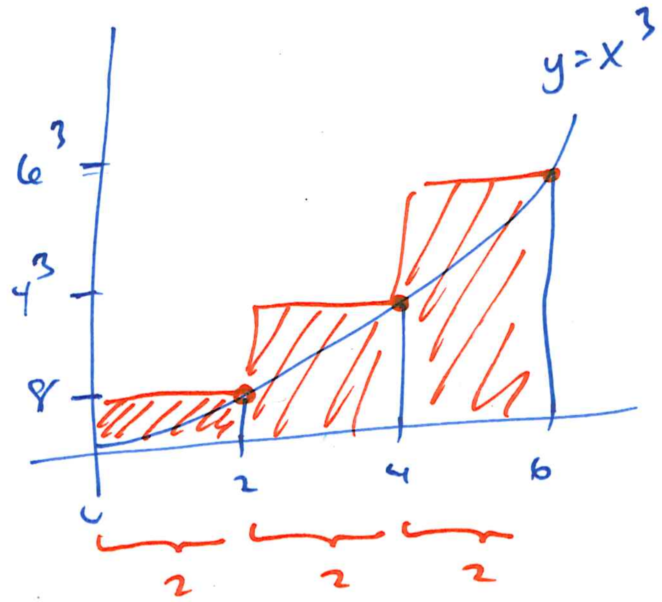
Grid points: 0, 2, 4, 6

Riemann Sum:

$$\Delta x\, f(x_1^*) + \Delta x\, f(x_2^*) + \Delta x\, f(x_3^*)$$

$$= 2 f(x_1^*) + 2 f(x_2^*) + 2 f(x_3^*)$$

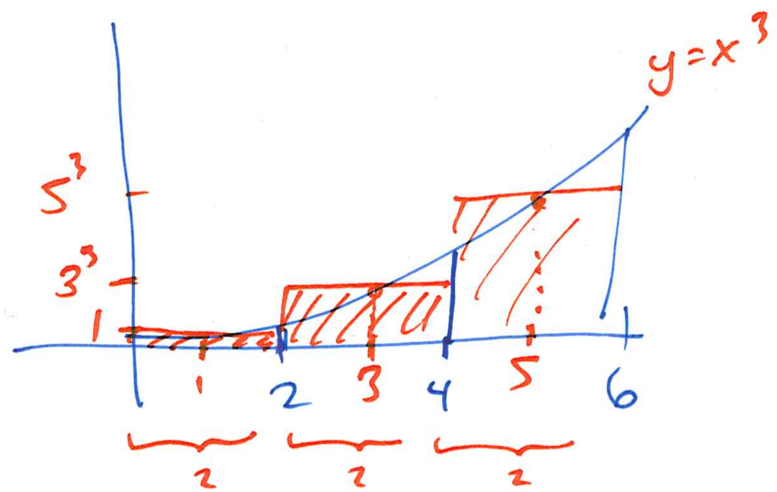
Right RS :



$$\boxed{2(8) + 2(4^3) + 2(6^3)}$$

Approximation of area under curve (Right RS)

Midpoint RS :



$$\boxed{(2)(1) + (2)(3^3) + (2)(5^3)}$$

Approx of area under curve:

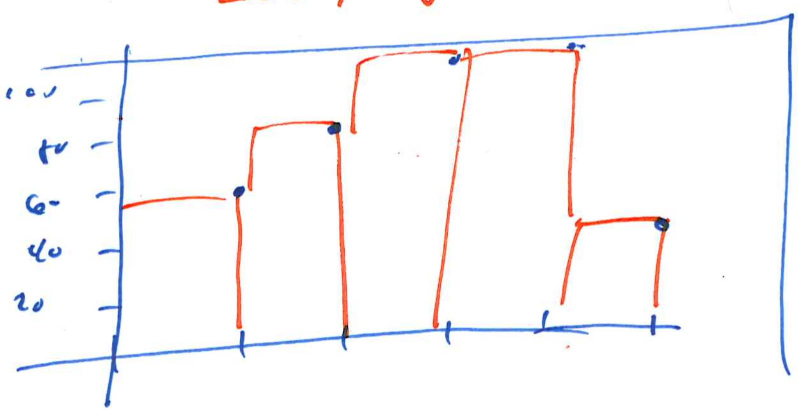
midpoint RS

(ex) Suppose a car's speed is:

time	12:00	12:15	12:30	12:45	1:00
speed	60 kph	80 kph	100 kph	100 kph	40 kph

How far did the car travel from 12:00 to 1:00?

Using Riemann Sum:
How many intervals? (n)
What is Δx?
Write out sum
Left, Right, MP



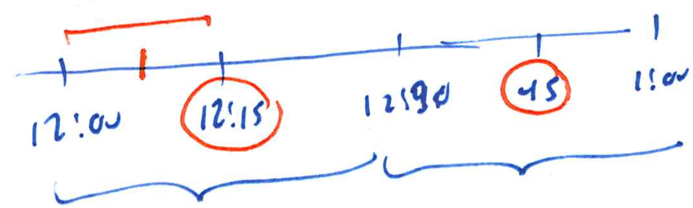
Left RS: $\Delta x = \frac{1}{4}$, $n=4$

$\left(\frac{1}{4}\right)(60) + \left(\frac{1}{4}\right)(80) + \left(\frac{1}{4}\right)(100) + \frac{1}{4}(100)$

Right RS: $\Delta x = \frac{1}{4}$, $n=4$

$\left(\frac{1}{4}\right)(80) + \frac{1}{4}(100) + \frac{1}{4}(100) + \frac{1}{4}(40)$

Midpoint RS: $n=2$, $\Delta x = \frac{1}{2}$



$\frac{1}{2}(80) + \frac{1}{2}(100)$

Quick Note:

The gradient of a function $f(x,y)$ is the vector

$$\nabla f = \langle f_x , f_y \rangle$$

ex $f(x,y) = x^2 + y^2$

$$\nabla f = \langle 2x, 2y \rangle$$

$$\nabla f = \lambda \nabla g$$

$$\Downarrow$$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases}$$

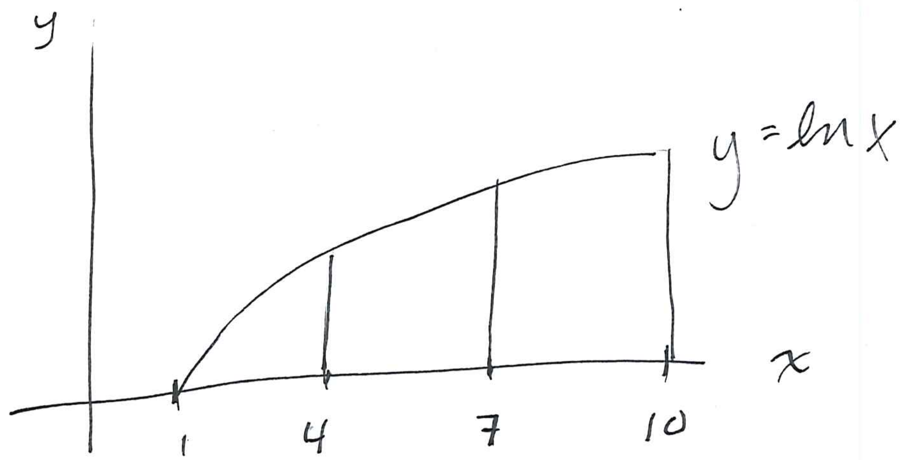
Riemann Sums

ex) Want to approximate area under $y = \ln x$

from $x = 1$ to $x = 10$

using Riemann Sums, 3 subintervals

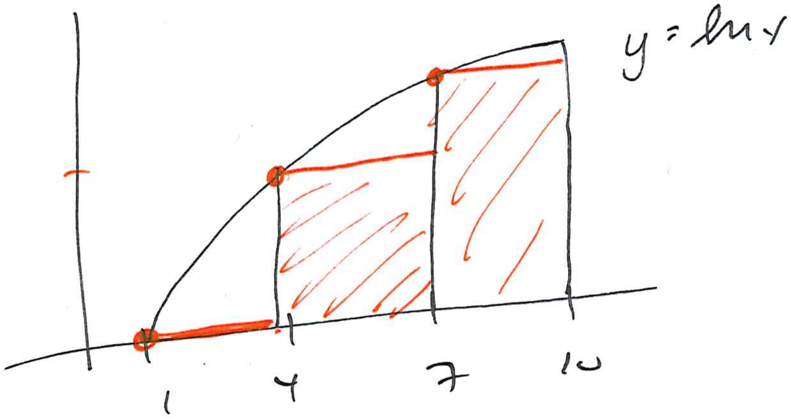
(3 rectangles)



Δx : width of each ▯

$$\Delta x = \frac{10-1}{3} = 3$$

Left:



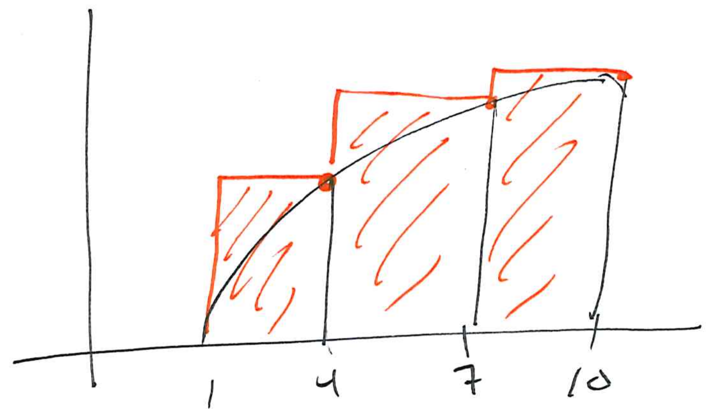
1st ▯: width 3
height 0

2nd ▯: width 3
height ln 4

3rd ▯: width 3
height ln 7

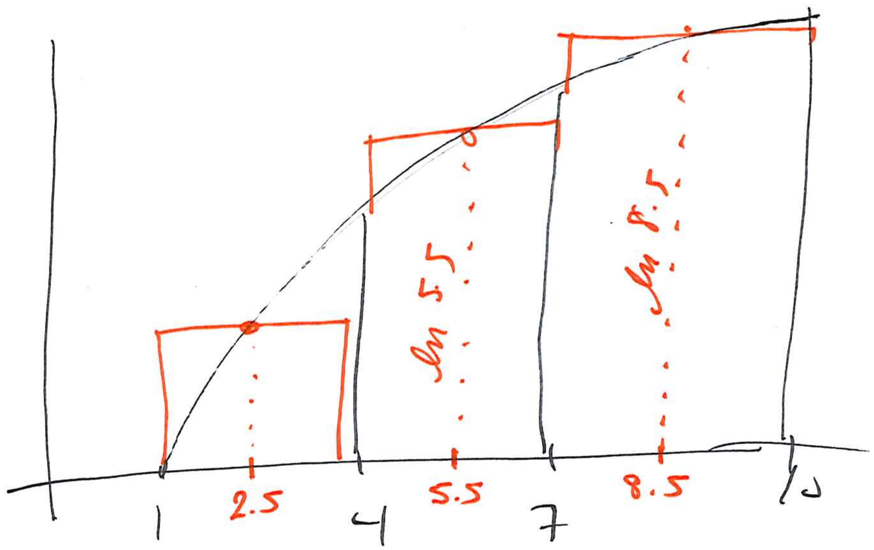
RS: $(3)(0) + (3)\ln 4 + (3)\ln 7$

Right:



RS: $(3)\ln 4 + (3)\ln 7 + 3\ln 10$

Midpoint



RS: $(3)\ln 2.5 + 3\ln 5.5 + 3\ln 8.5$

Review of Σ- notation

$$\sum_{k=a}^{b} f(k)$$

ex: $\displaystyle\sum_{k=-2}^{1} (2k+5) = \boxed{\underset{(k=-2)}{(-4+5)} + \underset{(k=-1)}{(-2+5)} + \underset{(k=0)}{(5)} + \underset{(k=1)}{(2+5)}}$

ex $\displaystyle\sum_{k=6}^{8} (k^2-k) = \boxed{(36-6)+(49-7)+(64-8)}$

Which are OK?

✓ (A) $\sum_{k=1}^{15}(k^2-k) = \sum_{k=1}^{15}k^2 - \sum_{k=1}^{15}k$

Order doesn't matter in addition

$1^2-1 + 2^2-2 + 3^2-3 \cdots$

$= (1^2+2^2+3^2+\cdots) - (1+2+3\cdots)$

✓ (B) $\sum_{k=1}^{15}3k = 3\sum_{k=1}^{15}k$ Factoring

$3(1)+3(2)+3(3)+\cdots+3(15)$

$= 3[1+2+3+\cdots+15]$

(C) $\sum_{k=1}^{15}k(k-1) \neq k\sum_{k=1}^{15}(k-1)$

$1(0) + 2(1) + 3(2) + \cdots$

✓ (D) $\sum_{k=1}^{15}(k-1) = -15 + \sum_{k=1}^{15}k$

$\|$

$(1-1)+(2-1)+(3-1)+(4-1)+\cdots+(15-1)$

$\|$

$\underbrace{(1+2+3+4+\cdots+15)}\ \underbrace{-1\ -1\ -1\ -1\ \cdots\ -1}_{15\text{ times}} = \sum_{k=1}^{15}k - \sum_{k=1}^{15}1$

$\|$

$\sum_{k=1}^{15}k - 15$

ex) Write in Σ-notation:

$2+3+4+5+6+7 = \sum_{k=2}^{7} k = \sum_{k=1}^{6} (k+1)$

$4+6+8+10+12 = \sum_{k=2}^{6} 2k$ $\quad \| \quad \sum_{k=2}^{6} k \cancel{+2} = 4 + \cancel{8}$

$k=2 \quad k=3$

$5+7+9+11+13 = \sum_{k=2}^{6} (2k+1)$

$3.5 + 6.5 + 9.5 + 12.5 + 15.5 = \sum_{k=1}^{5} \left(3k+\frac{1}{2}\right)$

$\frac{1}{2} + 1 + 2 + 4 + 8 + 16 + 32 = \sum_{k=-1}^{5} 2^k$

$2^{-1} \quad 2^0 \quad 2^1 \quad 2^2 \quad 2^3 \quad 2^4 \quad 2^5$

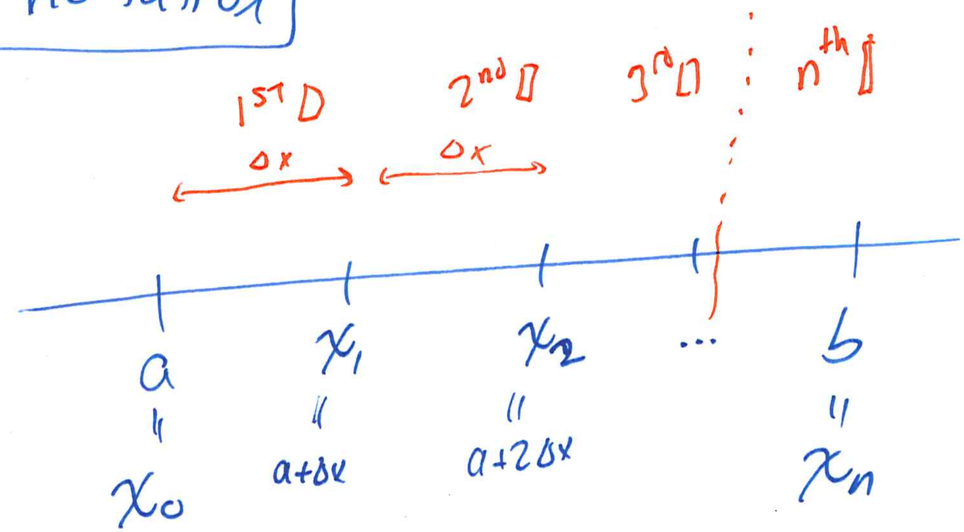
$-\frac{1}{2} + 1 - 2 + 4 - 8 + 16 - 32 = \sum_{k=-1}^{5} (-2)^k$

$(-2)^0 \quad (-2)^1 \quad (-2)^2 \quad (-2)^3$

$\sum_{k=2}^{6} (2k+1) = 5+7+9+11+13$

Riemann Sums in Σ-notation

function $f(x)$
over $[a,b]$
use n subintervals
(n rectangles)

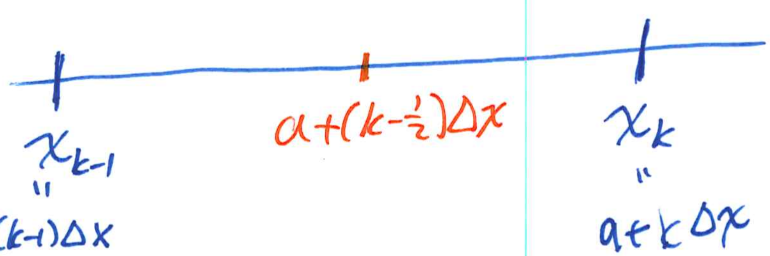


Bases of all rectangles:

$$\Delta x = \frac{b-a}{n}$$

Height: 1) depends on left/right/middle Riemann sum
2) depends on which □

(k^{th} □)



Area of k^{th} $\square$:

$(\text{base})(\text{height})$

$\Delta x\,(\text{height})$

Left: $\Delta x\, f(x_{k-1}) = \Delta x\, f(a+(k-1)\Delta x)$

Right: $\Delta x\, f(x_k) = \Delta x \cdot f(a+k\Delta x)$

Midpt: $\Delta x \cdot f\left(a+(k-\frac{1}{2})\Delta x\right)$

where: $\Delta x = \dfrac{b-a}{n}$

General Formulas for Riemann Sums:

Left: $\sum_{k=1}^{n} \Delta x \cdot f(a+(k-1)\Delta x)$ $\quad \Delta x = \frac{b-a}{n}$

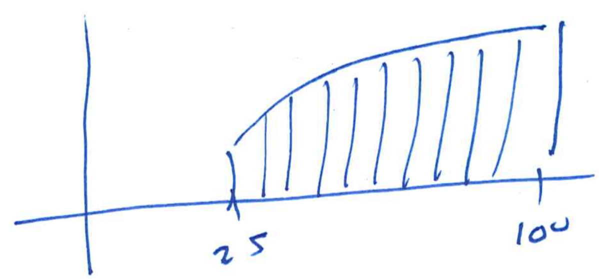
Right: $\sum_{k=1}^{n} \Delta x \cdot f(a+k\Delta x)$ $\quad \Delta x = \frac{b-a}{n}$

Midpoint: $\sum_{k=1}^{n} \Delta x \cdot f(a+(k-\frac{1}{2})\Delta x)$

(ex) $f(x) = \ln x$

$[25, 100]$

$n = 10$ rectangles



$\Delta x = \frac{100-25}{10} = \frac{75}{10} = 7.5$

$a = 25$

Left RS:

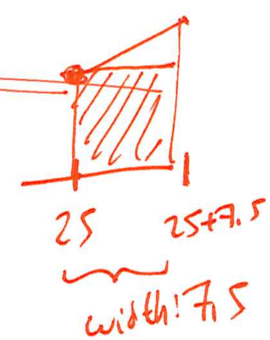
$$\sum_{k=1}^{10} \Delta x \cdot f(a+(k-1)\Delta x)$$

$$= \underbrace{\sum_{k=1}^{10} (7.5) \ln(25+(k-1)\cdot 7.5)}_{\text{some number}}$$

Right RS: $\underbrace{\sum_{k=1}^{10} (7.5) \cdot \ln(25+k\cdot 7.5)}_{\text{some number}}$

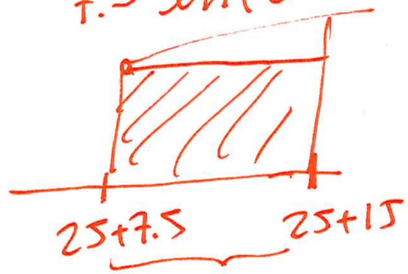
Midpt RS: $\underbrace{\sum_{k=1}^{10} (7.5) \ln(25+(k-\tfrac{1}{2})7.5)}_{\text{some number}}$

Left RS first terms: $(k=1)$ $7.5 \ln(25+0) = 7.5 \ln 25$



$(k=2)$ $7.5 \ln(25+7.5)$

area of 2^{nd} $\square$



etc...

Low-Degree Powers of k work nicely with Σ

p. 340 text

- $$\sum_{k=1}^{n} c = \underbrace{c + c + c + \cdots + c}_{n \text{ times}} = cn$$

  constant

  $k^0 = 1$

- $$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

- $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

- $$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

101

101

$$1 + 2 + 3 + \cdots + 49 + 50 + 51 + 52 + \cdots + 97 + 98 + 99 + 100$$

n

101

101

101

Sum: $101 \cdot 50 = (n+1)\left(\frac{n}{2}\right)$

(ex) Evaluate Rieman Sum (right)

$f(x) = x^2 + x$

$[1, 6]$

$n = 100$ rectangles

$\Delta x = \frac{6-1}{100} = \frac{5}{100} = \frac{1}{20}$

(width)

heigt: $\underset{\uparrow \text{ right endpt}}{f(x_k)} = f(a + k\Delta x) = f(1 + k \cdot \frac{1}{20})$

Sum: $\sum_{k=1}^{100} \underbrace{\frac{1}{20}}_{\substack{\Delta x \\ \text{width}}} \cdot \underbrace{f(\overbrace{1+\frac{k}{20}}^{x})}_{\substack{\text{height} \\ @ \\ \text{right} \\ \text{endpt}}} = \sum_{k=1}^{100} \frac{1}{20} \cdot \left[\left(1+\frac{k}{20}\right)^2 + \left(1+\frac{k}{20}\right) \right]$

$= \frac{1}{20} \sum_{k=1}^{100} \left(1 + \frac{2k}{20} + \frac{k^2}{20^2} + 1 + \frac{k}{20} \right) = \frac{1}{20} \sum_{k=1}^{100} \left(2 + \frac{3}{20}k + \frac{1}{20^2}k^2 \right)$

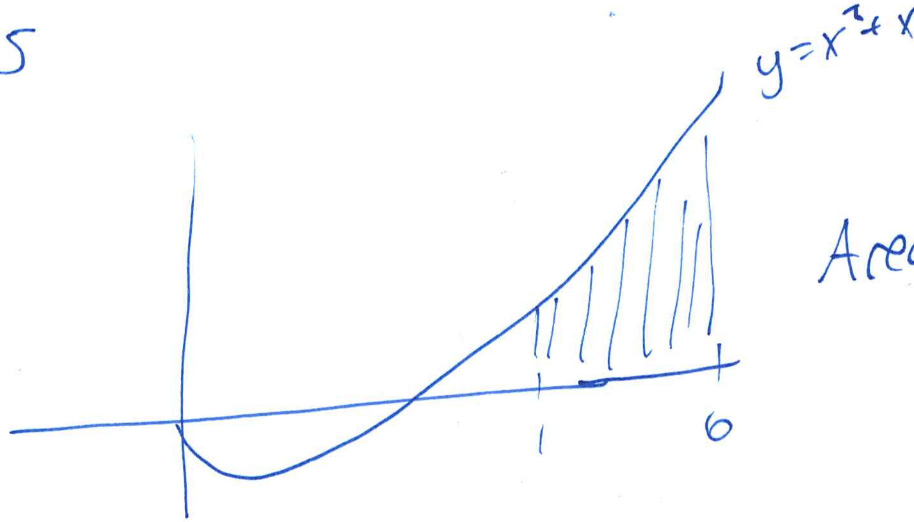
$$= \frac{1}{20}\left[\sum_{k=1}^{100} 2 + \sum_{k=1}^{100} \frac{3}{20}k + \sum_{k=1}^{100} \frac{1}{20^2}k^2\right]$$

$$= \frac{1}{20}\left[200 + \frac{3}{20}\sum_{k=1}^{100} k + \frac{1}{20^2}\sum_{k=1}^{100} k^2\right]$$

use list!

$$= \frac{1}{20}\left[200 + \frac{3}{20}\left(\frac{101\cdot 100}{2}\right) + \frac{1}{20^2}\left(\frac{100\cdot 101\cdot 201}{6}\right)\right.$$

$$= 90.16875$$



Area ≈ 90.16875

Riemann Sums

ex) Approx area under $f(x) = (x+2)^2$ on $[3, 5]$ using $n = 100$ rectangles.

Use right Riemann Sum

General formula for right RS: (n □, interval $[a, b]$)

$$\sum_{k=1}^{n} \Delta x \cdot f(a + k\Delta x) \quad \text{where } \Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{5-3}{100} = \frac{2}{100} = \frac{1}{50}$$

$a = 3 \quad n = 100$

Riemann Sum: $$\sum_{k=1}^{100} \frac{1}{50} f(\underbrace{3 + k(\tfrac{1}{50})}_{x}) = \sum_{k=1}^{100} \frac{1}{50}\left(2 + 3 + \frac{1}{50}k\right)^2$$

$f(x) = (x+2)^2$

$$= \sum_{k=1}^{100} \frac{1}{50}\left(5 + \frac{1}{50}k\right)^2 = \sum_{k=1}^{100} \frac{1}{50}\left(25 + \underbrace{2(5)\cdot\frac{1}{50}k}_{\frac{1}{5}k} + \frac{1}{50^2}k^2\right)$$

$$= \sum_{k=1}^{100} \left(\frac{1}{2} + \frac{1}{50 \cdot 5} k + \frac{1}{50^3} k^2 \right)$$

$$= \underbrace{\sum_{k=1}^{100} \frac{1}{2}} + \sum_{k=1}^{100} \frac{1}{50 \cdot 5} k + \sum_{k=1}^{100} \frac{1}{50^3} k^2$$

$$= 100\left(\tfrac{1}{2}\right) + \frac{1}{50 \cdot 5} \underbrace{\sum_{k=1}^{100} k}_{\text{formula}} + \frac{1}{50^3} \underbrace{\sum_{k=1}^{100} k^2}_{\text{formula}}$$

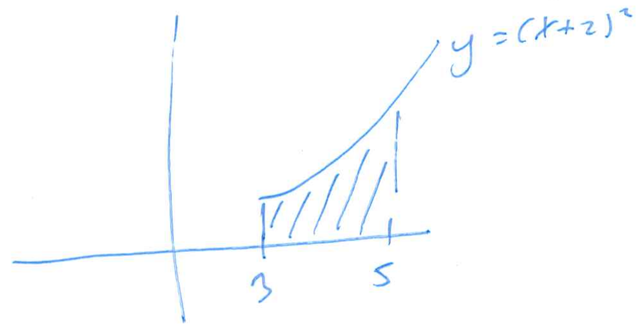
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= 50 + \frac{1}{50 \cdot 5} \cdot \left(\frac{100 \cdot 101}{2} \right) + \frac{1}{50^3} \cdot \left(\frac{100 \cdot 101 \cdot 201}{6} \right)$$

$$\underset{\text{(calculator)}}{=} \boxed{72.9068}$$

Approx (right RS) area:



ex) Area under $f(x) = (x+2)^2$, over $[3, 5]$

using $n = 100$ rectangles:

Midpoint RS

General formula for midpt RS:

$$\sum_{k=1}^{n\ (100)} \Delta x \cdot f\left(a + \left(k - \tfrac{1}{2}\right)\Delta x\right)$$

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{5-3}{100} = \frac{2}{100} = \frac{1}{50}$$

$$a = 3$$

$$\sum_{k=1}^{100} \frac{1}{50} \cdot f\left(3 + \left(k - \tfrac{1}{2}\right)\tfrac{1}{50}\right)$$

$$= \sum_{k=1}^{100} \frac{1}{50} f\left(3 - \frac{1}{100} + \frac{1}{50}k\right)$$

$$= \sum_{k=1}^{100} \frac{1}{50} \cdot f\left(\frac{299}{100} + \frac{1}{50}k\right)$$

$$= \sum_{k=1}^{100} \frac{1}{50} \cdot \left(2 + \frac{299}{100} + \frac{1}{50}k\right)^2$$

$$= \sum_{k=1}^{100} \frac{1}{50}\left(\frac{499}{100} + \frac{1}{50}k\right)^2$$

$$= \sum_{k=1}^{100} \frac{1}{50}\left(\left(\frac{499}{100}\right)^2 + 2\left(\frac{499}{100}\right)\cdot\frac{1}{50}k + \frac{1}{50^2}k^2\right)$$

$$= \sum_{k=1}^{100}\left(\frac{499^2}{100^2\cdot 50} + \frac{2\cdot 499}{100\cdot 50^2}k + \frac{1}{50^3}k^2\right)$$

$$= \underbrace{\sum_{k=1}^{100}\frac{499^2}{100^2\cdot 50}} + \sum_{k=1}^{100}\frac{2\cdot 499}{100\cdot 50^2}k + \sum_{k=1}^{100}\frac{1}{50^3}k^2$$

$$= \cancel{100}\left(\frac{499^2}{100^{\cancel{2}}\cdot 50}\right) + \frac{\cancel{2}\cdot 499}{\underset{50}{\cancel{100}}\cdot 50^2}\sum_{k=1}^{100}k + \frac{1}{50^3}\sum_{k=1}^{100}k^2$$

$$= \frac{499^2}{100\cdot 50} + \frac{499}{50^3}\sum_{k=1}^{100}k + \frac{1}{50^3}\sum_{k=1}^{100}k^2$$

$$= \frac{499^2}{100\cdot 50} + \frac{499}{50^3}\left(\frac{100\cdot 101}{2}\right) + \frac{1}{50^3}\cdot\left(\frac{100\cdot 101\cdot 201}{6}\right)$$

Formulas (p340)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad (n=100)$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

CALCULATOR: 72.6666

ex Find exact area under $y=(x+2)^2$; $[3,5]$

Plan:
- take RS using n rectangles ← might as well use easiest RS: right RS
- evaluate (in terms of n)
- take limit as $n \to \infty$

$\Delta x = \frac{b-a}{n} = \frac{2}{n}$

General form: $\sum_{k=1}^{n} \Delta x \, f(a + k\Delta x)$

$a = 3$

$$= \sum_{k=1}^{n} \left(\frac{2}{n}\right) \cdot f\left(3 + k \cdot \frac{2}{n}\right) = \sum_{k=1}^{n} \left(\frac{2}{n}\right)\left(2 + 3 + \frac{2}{n}k\right)^2$$

$$= \sum_{k=1}^{n} \left(\frac{2}{n}\right)\left(5 + \frac{2}{n}k\right)^2 = \sum_{k=1}^{n} \left(\frac{2}{n}\right)\left(25 + 2(5)\left(\frac{2}{n}k\right) + \left(\frac{2}{n}\right)^2 k^2\right)$$

$$= \sum_{k=1}^{n} \left(\frac{50}{n} + \frac{40}{n^2}k + \left(\frac{2}{n}\right)^3 k^2\right)$$

$$= \sum_{k=1}^{n} \frac{50}{n} + \sum_{k=1}^{n} \frac{40}{n^2} k + \sum_{k=1}^{n} \frac{8}{n^3} k^2$$

$$= \sum_{k=1}^{n} \frac{50}{n} + \frac{40}{n^2} \sum_{k=1}^{n} k + \frac{8}{n^3} \sum_{k=1}^{n} k^2$$

$$= (\cancel{n})\left(\frac{50}{\cancel{n}}\right) + \frac{\cancel{40}^{20}}{\cancel{n^2}} \cdot \frac{\cancel{n}(n+1)}{\cancel{2}} + \frac{\cancel{8}^{4}}{\cancel{n^3}_{2}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}_{3}}$$

$$= 50 + 20 \cdot \left(\frac{n+1}{n}\right) + \frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)$$

$$= 50 + 20\left(1 + \frac{1}{n}\right) + \frac{4}{3}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= 50 + 20\left(1 + \frac{1}{n}\right) + \frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

$$\xrightarrow[n \to \infty]{} 50 + 20(1+0) + \frac{4}{3}(1+0)(2+0)$$

$$= 50 + 20 + \frac{8}{3}$$

$$= 70 + 8/3$$

$$= 70 + 2 + 2/3 = \boxed{72 + 2/3} = 72.\overline{66}$$

Formulas: (p 340)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

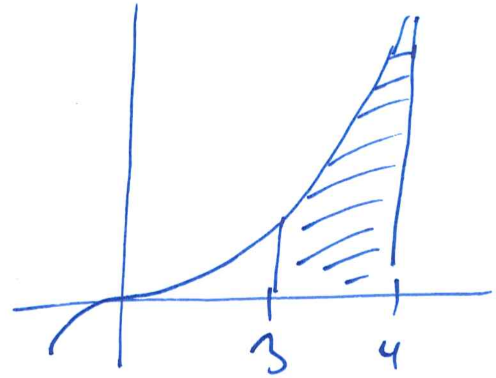
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

exact area under $y = (x+2)^2$ from $x=3$ to $x=5$

ex Find the exact area under the curve

$y = x^3$

over the interval $[3,4]$



$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

General Right Riemann Sum:

$$\sum_{k=1}^{n} \Delta x \, f(a + k\Delta x)$$

$$\Delta x = \frac{b-a}{n} = \frac{4-3}{n} = \frac{1}{n}$$

formulas:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=1}^{n} \frac{1}{n} f\left(3 + k\left(\frac{1}{n}\right)\right)$$

$$= \sum_{k=1}^{n} \frac{1}{n} \cdot \left(3 + \frac{k}{n}\right)^3$$

$$= \sum_{k=1}^{n} \frac{1}{n} \left[27 + 3 \cdot 9 \cdot \frac{k}{n} + 3 \cdot 3 \cdot \left(\frac{k}{n}\right)^2 + \left(\frac{k}{n}\right)^3\right]$$

$$= \sum_{k=1}^{n} \frac{1}{n}\left[27 + \frac{27}{n}k + \frac{9}{n^2}k^2 + \frac{1}{n^3}k^3\right]$$

$$= \sum_{k=1}^{n}\left(\frac{27}{n} + \frac{27}{n^2}k + \frac{9}{n^3}k^2 + \frac{1}{n^4}k^3\right)$$

$$= \sum_{k=1}^{n}\frac{27}{n} + \sum_{k=1}^{n}\frac{27}{n^2}k + \sum_{k=1}^{n}\frac{9}{n^3}k^2 + \sum_{k=1}^{n}\frac{1}{n^4}k^3$$

FORMULAS

$$= n\left(\frac{27}{n}\right) + \frac{27}{n^2}\sum_{k=1}^{n}k + \frac{9}{n^3}\sum_{k=1}^{n}k^2 + \frac{1}{n^4}\sum_{k=1}^{n}k^3$$

$$= 27 + \frac{27}{n^{\cancel{2}}}\left(\frac{\cancel{n}(n+1)}{2}\right) + \frac{\cancel{9}\,3}{n^{\cancel{3}\,2}}\left(\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}\,2}\right) + \frac{1}{n^{\cancel{4}\,2}}\left(\frac{\cancel{n^2}(n+1)^2}{4}\right)$$

$$= 27 + \frac{27}{2}\left(\frac{n+1}{n}\right) + \frac{3}{2}\left(\frac{(n+1)}{n}\cdot\frac{(2n+1)}{n}\right) + \frac{1}{4}\left(\frac{n+1}{n}\right)^2$$

$$= 27 + \frac{27}{2}\left(1+\frac{1}{n}\right) + \frac{3}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{1}{4}\left(1+\frac{1}{n}\right)^2$$

$$\xrightarrow[n\to\infty]{} 27 + \frac{27}{2}(1+0) + \frac{3}{2}(1+0)(2+0) + \frac{1}{4}(1+0)^2$$

$$= 27 + \frac{27}{2} + 3 + \frac{1}{4} = 30 + \frac{27}{2} + \frac{1}{4} = 30 + 13 + \frac{1}{2} + \frac{1}{4} = 43 + \frac{3}{4}$$

$$= \boxed{43.75}$$

To find exact area under $y = f(x)$, $[a,b]$:

$$\text{Area} = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \Delta x \, f(x_k^*) \right)$$

could be left / right / MP

$$= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \Delta x \, f(a + k\Delta x) \right)$$

Last time, we said:

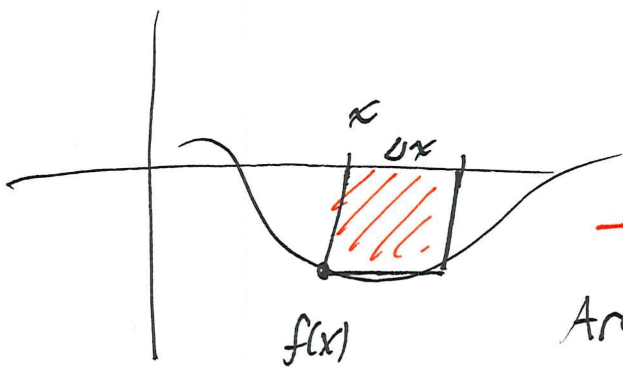
Area under curve $y = f(x)$, $[a, b]$ is

$$\lim_{n \to \infty} \sum_{k=1}^{n} \Delta x \cdot f(x_k^*) \quad , \quad \Delta x = \frac{b-a}{n}$$

x_k^* is between

$a + (k-1)\Delta x$ + $a + k\Delta x$

Q: What if $f(x) < 0$?



− Area

Area: (width)(height)

$= \Delta x \, |f(x)|$

$= \Delta x \, (-f(x))$

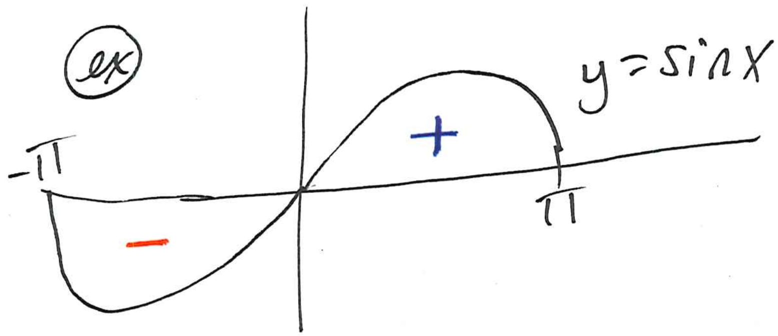
$= -\Delta x \, f(x)$

Actually calculating: "net area"

Area above axis

− area below axis

ex



$f(x) = \sin x$

$[-\pi, \pi]$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Delta x \cdot f(x_k^*) = 0$$

(positive area exactly cancels out negative area)

Notation: Definite Integrals

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x \cdot f(x_k^*), \quad \text{where } \Delta x = \frac{b-a}{n} \text{ etc.}$$

dx : "differential"
$\lim_{n\to\infty} \Delta x$

a, b : bounds
w/o bounds, integral is "indefinite"

$\int$ "integral sign"
elongated S for "sum"
same as Σ

Properties of Definit Integral

1. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

why? $\Delta x = \frac{b-a}{n} = -\left[\frac{a-b}{n}\right]$

2. $$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

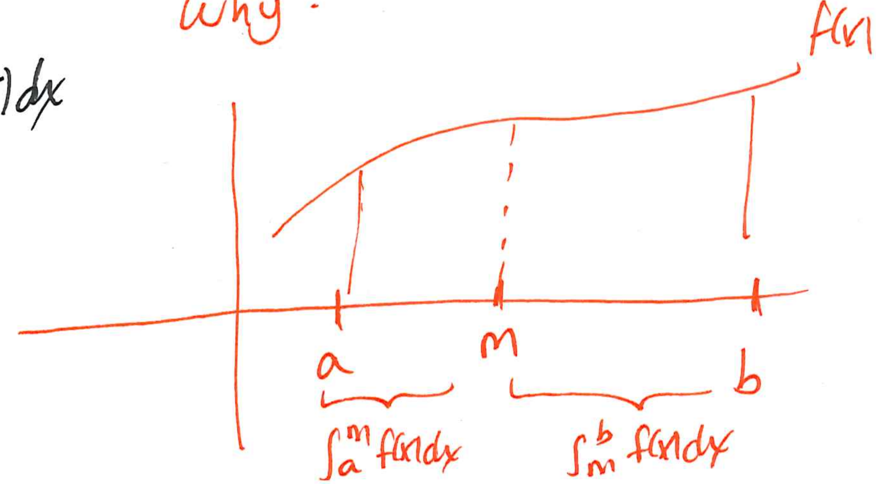
why? $\sum_{k=1}^{n}(f+g) = \sum_{k=1}^{n} f + \sum_{k=1}^{n} g$

3. $$\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$$

c - constant

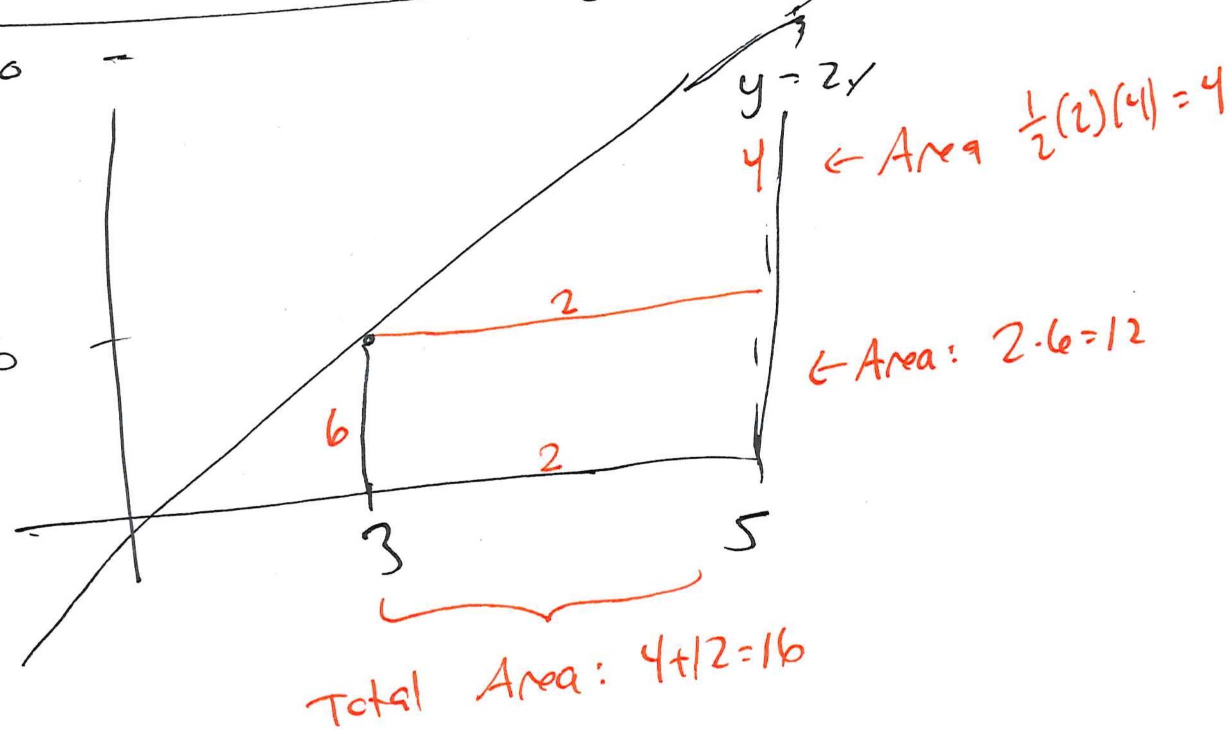
4. $$\int_{\underline{a}}^{\underline{b}} f(x)\,dx = \int_{\underline{a}}^{m} f(x)\,dx + \int_m^{\underline{b}} f(x)\,dx$$

why?

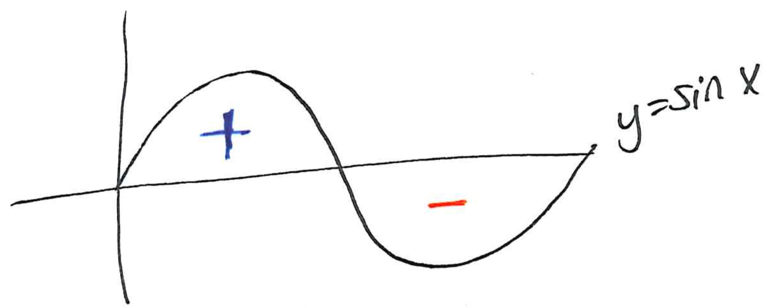


Evaluating Definite Integrals Using Geometry

ex $\int_3^5 2x\,dx = 16$



ex $\int_0^{2\pi} \sin x\,dx = 0$



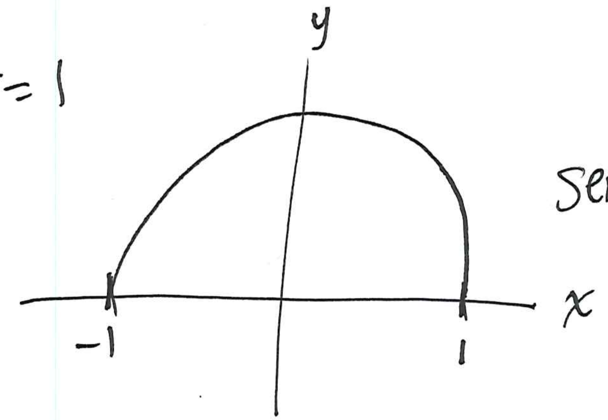
(ex) $\int_{-1}^{1} \sqrt{1-x^2}\,dx = \pi/2$

$y = \sqrt{1-x^2}$

$y^2 = 1-x^2$

$x^2+y^2 = 1$

$y \geq 0$

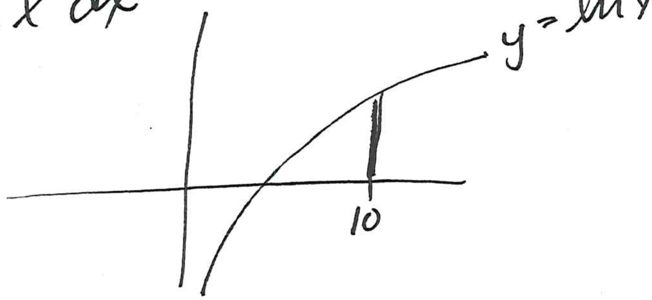


semicircle

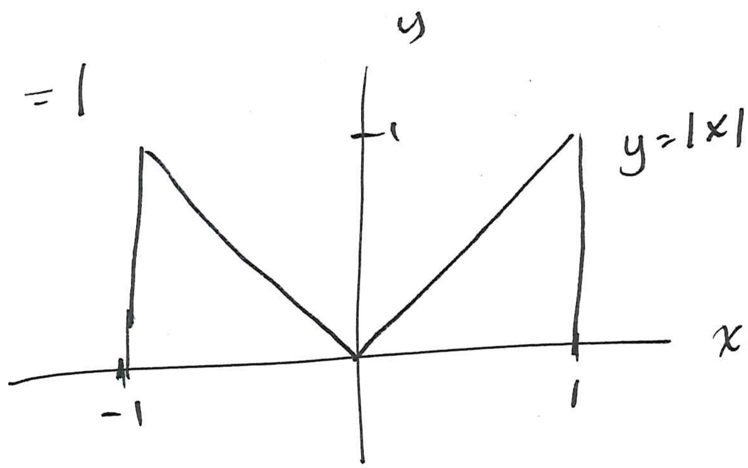
$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(1) = \pi/2$

(ex) $\int_{10}^{10} \ln x\,dx = 0$

no width!



(ex) $\int_{-1}^{1} |x|\,dx = 1$

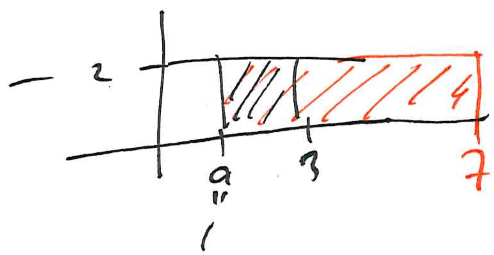


Ch 5.3 Fundamental Theorem of Calculus

Area function: $A(x) = \int_a^x f(t)\,dt$

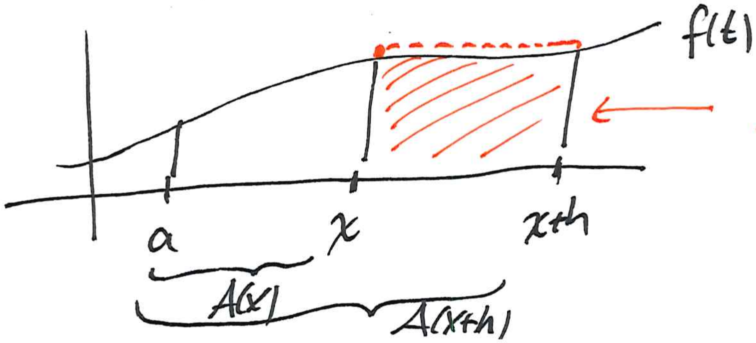
ex) If $f(t) = 2$ $\quad a = 1$

$A(3) = 4$ $\qquad A(7) = 12$



Derivative of Area Function:

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} \frac{h \cdot f(x)}{h} = f(x)$$



$A(x+h) - A(x) \approx h \cdot f(x)$

As $h \to 0$, $A(x+h) - A(x) \longrightarrow h \cdot f(x)$

Fundamental Theorem of Calculus, Part 1:

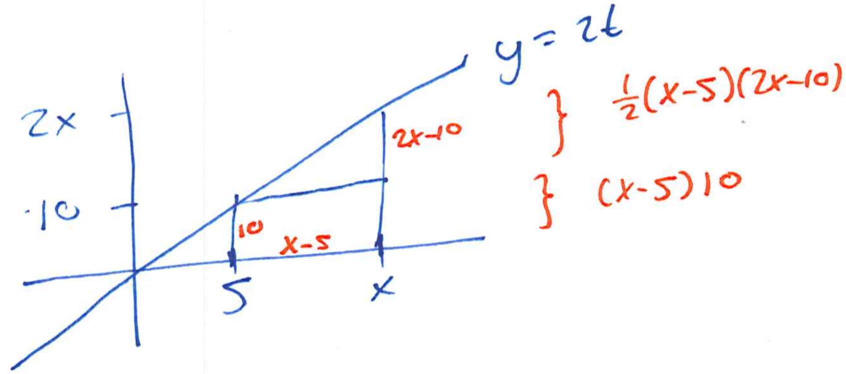
If f is continuous on $[a,b]$, then the area function

$A(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$

and differentiable on (a,b), and

$$A'(x) = f(x)$$

ex $A(x) = \int_5^x 2t\,dt$



$$\begin{aligned}
&= 10(x-5) + \tfrac{1}{2}(x-5)(2x-10) \\
&= 10(x-5) + (x-5)(x-5) \\
&= (x-5)(10+x-5) \\
&= (x-5)(x+5) \\
&= x^2-25
\end{aligned}$$

NOTICE: $A(x) = x^2-25$

$A'(x) = 2x = f(x)$

We are considering $A(x) = \int_p^x f(t)\,dt$

FTC(I): $A'(x) = f(x)$

<u>Lots</u> of functions have $f(x)$ as derivative.

Take <u>any</u> function $F(x)$ such that $F'(x) = f(x)$.

Since A, F have same derivative, they only differ by some constant, say C:

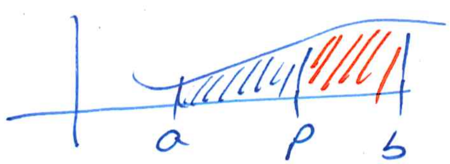
$$F(x) = A(x) + C$$

$$F(b) - F(a) = [A(b) + \cancel{C}] - [A(a) + \cancel{C}] = A(b) - A(a)$$

Notice:

$$= \int_p^b f(t)\,dt - \int_p^a f(t)\,dt = \int_p^b f(t)\,dt + \int_a^p f(t)\,dt$$

$$= \int_a^p f(t)\,dt + \int_p^b f(t)\,dt = \int_a^b f(t)\,dt$$



Fundamental Theorem of Calculus, Part II

If f is continuous on $[a,b]$, and F is any antiderivative of f (I mean: $F'(x) = f(x)$), then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

ex $\int_5^{10} 2x\,dx = 10^2 - 5^2 = \boxed{75}$

could use geometry
could also use FTC

$f(x) = 2x$

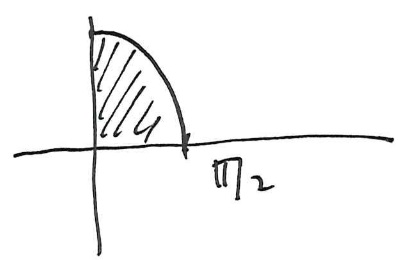
$F(x) = x^2$

ex $\int_0^{\pi/2} \cos x\,dx = \sin(\pi/2) - \sin(0)$

$= 1 - 0 = \boxed{1}$

$f(x) = \cos x$

$F(x) = \sin x$



$$\int \cos x \, dx = \sin x + C$$

$$\int 15 \cos x \, dx = 15 \sin x + C$$

Constant Powers of x:

$f(x)$	$f'(x)$
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^n	nx^{n-1}

$f(x)$	$\int f(x)\,dx$
1	$x + C$
x	$\frac{1}{2}x^2 + C$
x^2	$\frac{1}{3}x^3 + C$
x^3	$\frac{1}{4}x^4 + C$
x^n	$\frac{1}{n+1}x^{n+1} + C$ if $n \neq -1$
$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert + C$

$$\int \sqrt{x}\, dx = \int x^{1/2} dx = \frac{2}{3} x^{3/2} + C$$

indefinite (no bounds)

$$\int_1^9 \sqrt{x}\, dx = \frac{2}{3}(9)^{3/2} - \frac{2}{3}(1)^{3/2} = \frac{2}{3} \cdot \sqrt{9}^{\,3} - \frac{2}{3} = \frac{2}{3} \cdot 27 - 2/3$$

$$= \frac{2}{3}(26)$$

definite integral - area

ex $\frac{d}{dx}\{\arctan x\} = \frac{1}{1+x^2}$, so: $\int \frac{1}{1+x^2}dx = \arctan x + C$

Note: This is a little surprising:

$$\int \frac{1}{x^2}dx = \int x^{-2}dx = -x^{-1} + C = \frac{-1}{x} + C$$

$$\int \frac{1}{x}dx = \ln|x| + C$$

$$\frac{d}{dx}\left\{\frac{1}{a}\arctan\left(\frac{x}{a}\right)\right\} = \frac{1}{a}\cdot\frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a} = \frac{1}{a^2\left(1+\frac{x^2}{a^2}\right)} = \frac{1}{a^2+x^2}$$

(a constant)

So: $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

ex) $\frac{d}{dx}\{\arcsin x\} = \frac{1}{\sqrt{1-x^2}}$

$$\frac{d}{dx}\{\arcsin(\tfrac{x}{a})\} = \frac{1}{\sqrt{1-(\frac{x}{a})^2}} \cdot \frac{1}{a} = \frac{1}{\sqrt{a^2}\cdot\sqrt{1-\frac{x^2}{a^2}}}$$

(a>0)

$$= \frac{1}{\sqrt{a^2-\cancel{a^2}\cdot\frac{x^2}{\cancel{a^2}}}} = \frac{1}{\sqrt{a^2-x^2}}$$

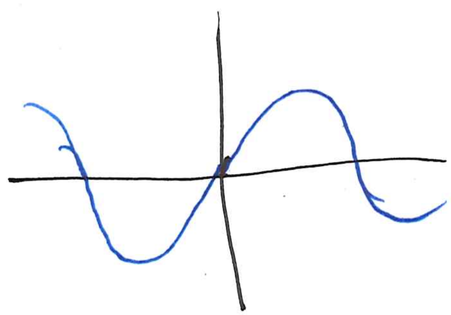
So: $\boxed{\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin(\tfrac{x}{a}) + C}$

Even + Odd Functions

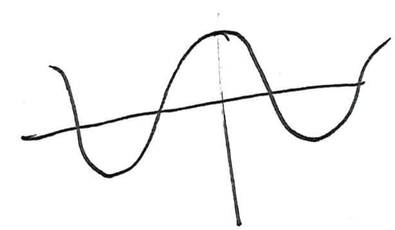
Odd function: $f(-x) = -f(x)$

ex: $f(x) = \sin x$

$f(x) = x^3 \longrightarrow f(-2) = -8 = -f(2)$



Odd Function: $\int_{-a}^{a} f(x)dx = 0$



Even function: $f(-x) = f(x)$

ex: $\cos x$

$f(x) = x^2$ $\quad f(-2) = 4 = f(2)$

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

Substitution Rule

(Chain Rule in reverse)

ex $f(x) = \sin(3x^2+x)$

$f'(x) = \cos(3x^2+x)\cdot(6x+1)$

$$\int \cos(\overbrace{3x^2+x}^{\text{inside}})\cdot(\overbrace{6x+1}^{\text{deriv. of inside}})\,dx =$$

$$\sin(3x^2+x) + C$$

ex Chain rule:

$$\frac{d}{dx}\{f(g(x))\} = f'(g(x))\cdot g'(x)$$

Backwards:

$$\int f'(g(x))\cdot g'(x)\,dx = f(g(x)) + C$$

Mnemonic: change of variable

"dictionary"

$\boxed{g(x) = u}$

$\frac{du}{dx} = g'(x)$

$\boxed{du = g'(x)\,dx}$

$$\int f'(g(x))\cdot \underbrace{g'(x)\,dx} =$$

$$\int f'(u)\cdot du = f(u) + C$$

$$= f(g(x)) + C$$

ex $\displaystyle\int e^{\boxed{\sin x}} \underbrace{\cos x\, dx}_{du} = \int e^u\, du = e^u + C = \boxed{e^{\sin x} + C}$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\, dx$

Check: $\frac{d}{dx}\left\{ e^{\sin x} + C \right\} = e^{\sin x} \cdot \cos x$ ✓

ex $\displaystyle\int e^x \sin(e^x)\, dx = \int \sin u\, du = -\cos u + C = \boxed{-\cos(e^x) + C}$

$u = e^x$

$\frac{du}{dx} = e^x$

$du = e^x\, dx$

(ex) $$\int \frac{e^x}{e^x+15}\,dx = \int \frac{1}{u}\cdot du = \ln|u|+C$$

$$= \ln|e^x+15|+C$$

$$= \boxed{\ln(e^x+15)+C}$$

$u = e^x+15$

$\frac{du}{dx} = e^x$

$du = e^x dx$

(ex) $$\int x\sec(x^2)\tan(x^2)\,dx = \int \frac{1}{2}\sec u\cdot\tan u\,du$$

$$= \frac{1}{2}\sec u + C = \boxed{\frac{1}{2}\sec(x^2)+C}$$

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

ex $\int \sin x \underbrace{\cos x\, dx}_{du} = \int u \cdot du = \frac{1}{2}u^2 + C$

$= \boxed{\frac{1}{2}(\sin x)^2 + C}$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\, dx$

ex $\int x^4 \underbrace{(x^5+1)}^8 \, dx = \int \frac{1}{5} \cdot u^8 \, du = \frac{1}{5} \cdot \frac{1}{9} u^9 + C$

$u = x^5 + 1$

$\frac{du}{dx} = 5x^4$

$du = 5x^4 \, dx$

$\frac{1}{5} du = x^4 \, dx$

$= \frac{1}{45} u^9 + C$

$= \boxed{\frac{1}{45}(x^5+1)^9 + C}$

ex $\displaystyle\int \frac{s}{s-3}\,ds = \int \frac{u+3}{u}\,du = \int \frac{u}{u}+\frac{3}{u}\,du$

$\displaystyle = \int\left(1+\frac{3}{u}\right)du = u+3\ln|u|+C$

$= \boxed{s-3+3\ln|s-3|+C}$

$u = s-3$

$du = ds$

$s = u+3$

ex $\displaystyle\int \frac{\sec^2(\sqrt{x}+1)}{\sqrt{x}}\,dx = \int 2\sec^2 u\,du = 2\tan u + C$

$= \boxed{2\tan(\sqrt{x}+1)+C}$

$u = \sqrt{x}+1$

$\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$

$du = \dfrac{1}{2\sqrt{x}}\,dx$

$2\,du = \dfrac{1}{\sqrt{x}}\,dx$

ex $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int -\frac{1}{u} du$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$= -\ln|u| + C$

$= -\ln|\cos x| + C$

$= \ln|(\cos x)^{-1}| + C$

$= \ln\left|\frac{1}{\cos x}\right| + C$

$= \boxed{\ln|\sec x| + C}$

*memorize

ex $\displaystyle\int x^5\sqrt{x^3+1}\,dx = \int \underbrace{x^3}_{u-1}\underbrace{\sqrt{x^3+1}}_{\sqrt{u}}\underbrace{x^2dx}_{\frac{1}{3}du}$

$u = x^3+1 \rightarrow x^3 = u-1$

$\frac{du}{dx} = 3x^2$

$du = 3x^2dx$

$\frac{1}{3}du = x^2dx$

$$= \int \tfrac{1}{3}(u-1)\sqrt{u}\,du$$

$$= \int \tfrac{1}{3}(u-1)u^{1/2}\,du$$

$$= \tfrac{1}{3}\int \left(u^{3/2} - u^{1/2}\right) du$$

$$= \tfrac{1}{3}\left[\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right] + C$$

$$= \tfrac{2}{15}u^{5/2} - \tfrac{2}{9}u^{3/2} + C$$

$$= \boxed{\tfrac{2}{15}(x^3+1)^{5/2} - \tfrac{2}{9}(x^3+1)^{3/2} + C}$$

ex $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos x \, dx}{\sin^3 x} = \int_{1/\sqrt{2}}^{1} \frac{1}{u^3}\, du = \int_{1/\sqrt{2}}^{1} u^{-3}\, du =$

$u = \sin x$
$du = \cos x \, dx$

If $x = \pi/4$, $u = \sin(\pi/4) = 1/\sqrt{2}$
If $x = \pi/2$, $u = \sin(\pi/2) = 1$

$$\left.\frac{u^{-2}}{-2}\right|_{1/\sqrt{2}}^{1} =$$

$$\left(\frac{1^{-2}}{-2}\right) - \left(\frac{(1/\sqrt{2})^{-2}}{-2}\right)$$

$$= \frac{-1}{2} + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^{-2}$$

$$= \frac{-1}{2} + \frac{1}{2}(\sqrt{2})^2 = \frac{-1}{2} + \frac{1}{2}(2) = 1 - 1/2$$

$$= 1/2$$

(ex) $\displaystyle\int_0^2 \frac{2s}{s^2+1}\,ds = \int_1^5 \frac{1}{u}\,du = \ln|u|\Big|_1^5 = \ln 5 - \cancel{\ln 1}$

$= \boxed{\ln 5}$

$u = s^2+1$

$\frac{du}{ds} = 2s$

$du = 2s\,ds$

If $s=0$, $u = 0^2+1 = 1$

If $s=2$, $u = 2^2+1 = 5$

(ex) $\displaystyle\int_5^{10} \frac{8t+6}{2t^2+3t}\,dt = \int_{65}^{230} \frac{2}{u}\,du$

If $t=5$, $u = 2(5)^2+3\cdot 5 = 50+15 = 65$

If $t=10$, $u = 2\cdot 10^2+3\cdot 10 = 200+30 = 230$

$u = 2t^2+3t$

$\frac{du}{dt} = 4t+3$

$du = (4t+3)\,dt$

$2du = (8t+6)\,dt$

$= 2\ln|u|\Big|_{65}^{230} = 2\ln(230) - 2\ln(65)$

$= \boxed{2\ln\left(\frac{230}{65}\right)}$

(ex) $\int e^{x+e^x}\,dx = \int \underbrace{e^x e^{e^x}\,dx} = \int du = u + C = \boxed{e^{e^x} + C}$

$u = e^{(e^x)}$

$\frac{du}{dx} = e^{(e^x)} \cdot e^x$

$du = e^{e^x} e^x\,dx$

Check:

$\frac{d}{dx}\left\{ e^{e^x} + C \right\} = e^{e^x} \cdot e^x$ ✓

(ex) $\int \frac{dx}{e^x + e^{-x}} \left(\frac{e^x}{e^x}\right) = \int \frac{e^x\,dx}{(e^x)^2 + 1} = \int \frac{1}{u^2+1}\,du = \arctan u + C$

du

$= \boxed{\arctan(e^x) + C}$

$u = e^x$

$du = e^x\,dx$

Ch 7.2: Integration By Parts

(product rule – backwards)

$$\frac{d}{dx}\{u(x)\cdot v(x)\} = u'(x)\,v(x) + u(x)\cdot v'(x)$$

So: $$\int [u'(x)v(x) + u(x)v'(x)]\,dx = u(x)\cdot v(x) + C$$

$$\int u'(x)v(x)\,dx + \underbrace{\int u(x)v'(x)\,dx} = u(x)\,v(x) + C$$

So: $$\int u(x)\cdot v'(x)\,dx = u(x)\,v(x) - \int u'(x)\,v(x)\,dx + C$$

Mnemonic: $$\boxed{\int u\,dv = uv - \int v\,du}$$

*memorize

ex $\int x \sin x\, dx = -x\cos x - \int -\cos x\,(1)\,dx =$

$-x\cos x + \int \cos x\, dx$

$u: x \qquad du: 1\,dx$

$dv: \sin x\,dx \qquad \underbrace{v: -\cos x}$

$= \boxed{-x\cos x + \sin x + C}$

$\int u\,dv = uv - \int v\,du$

ex $\int x \underline{\ln x}\, dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\, dx$

$u: \ln x \qquad du: \frac{1}{x}\, dx$

$dv: x\,dx \qquad v: \frac{1}{2}x^2$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\, dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\left(\frac{1}{2}x^2\right) + C$

$= \boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

ex $\int (x+1)\sec^2 x\,dx = (x+1)\tan - \int \tan x \cdot dx$

$u : x+1$ $\quad du : 1\,dx$

$dv : \sec^2 x\,dx$ $\quad v : \tan x$

$$= (x+1)\tan x - \int \frac{\sin x}{\cos x}\,dx$$

$s = \cos x$

$ds = -\sin x\,dx$

$$= (x+1)\tan x - \int \frac{-ds}{s}$$

$$= (x+1)\tan x + \ln|s| + C$$

$$= \boxed{(x+1)\tan x + \ln|\cos x| + C}$$

ex) $\int \underbrace{x}\,\underbrace{e^{6x}}\,dx = \frac{x}{6}e^{6x} - \int \frac{1}{6}e^{6x}\,dx$

$u: x \qquad du: 1\cdot dx$

$dv: e^{6x}dx \qquad v: \frac{1}{6}e^{6x}$

$= \frac{x}{6}e^{6x} - \frac{1}{6}\left(\frac{1}{6}e^{6x}\right) + C$

$= \boxed{\frac{1}{6}e^{6x}\left(x - \frac{1}{6}\right) + C}$

$\boxed{\int u\,dv = uv - \int v\,du}$

ex) $\int (3t+5)\cos\left(\frac{t}{4}\right)dt =$

$(3t+5)(4\sin(\frac{t}{4}))$

$- \int 12\sin(\frac{t}{4})dt$

$u: 3t+5 \qquad du: 3\,dt$

$dv: \cos(\frac{t}{4})dt \qquad v: 4\sin(\frac{t}{4})$

$= (3t+5)(4\sin(\frac{t}{4})) - 12\cdot(4)\left[-\cos(\frac{t}{4})\right] + C$

$= \boxed{4(3t+5)\sin\frac{t}{4} + 48\cos(\frac{t}{4}) + C}$

$v = \int dv = \int \cos(\frac{t}{4})dt = \int \cos(u)\cdot 4\,du$

$u = \frac{t}{4}$

$du = \frac{1}{4}dt$

$4du = dt$

$= 4\sin u$

$= \boxed{4\sin(\frac{t}{4})}$

(ex) $\int x^3 \ln x\, dx = \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\, dx$

$u: \ln x \qquad du: \frac{1}{x}dx$

$dv: x^3 dx \qquad v: \frac{1}{4}x^4$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\left(\frac{1}{4}x^4\right) + C$

$= \boxed{\frac{1}{4}x^4\left(\ln x - \frac{1}{4}\right) + C}$

(ex) $\int x^2 \ln^2 x\, dx$

$u: \ln^2 x \qquad du: 2\ln x \cdot \frac{1}{x}\, dx$

$dv: x^2 dx \qquad v: \frac{1}{3}x^3$

$= \frac{1}{3}x^3 \ln^2 x - \int \frac{1}{3}x^3 \cdot 2\ln x \cdot \frac{1}{x} dx$

$= \frac{1}{3}x^3 \ln^2 x - \frac{2}{3}\int x^2 \ln x\, dx =$

$u: \ln x \qquad du = \frac{1}{x}dx$

$dv: x^2 dx \qquad v = \frac{1}{3}x^3$

$\frac{1}{3}x^3 \ln^2 x - \frac{2}{3}\left[\frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \cdot \frac{1}{x} dx\right]$

$= \frac{1}{3}x^3 \ln^2 x - \frac{2}{3}\left[\frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx\right]$

$= \frac{1}{3}x^3 \ln^2 x - \frac{2}{3}\left[\frac{1}{3}x^3 \ln x - \frac{1}{3}\cdot\frac{1}{3}x^3\right] + C$

ex $\int \ln x \, dx = x \ln x - \int x \left(\frac{1}{x}\right) dx$

$u : \ln x \qquad du : \frac{1}{x} dx$

$dv : 1\, dx \qquad v : x$

$= x \ln x - \int 1 \, dx$

$= \boxed{x \ln x - x + C}$

ex $\int \underbrace{\arctan x}\, \underbrace{dx} = x \arctan x - \int \frac{x}{1+x^2} dx =$

$\frac{1}{2} dw$

$u : \arctan x \qquad du : \frac{1}{1+x^2} dx$

$dv : 1 \cdot dx \qquad v = x$

Substitution:

$w = 1+x^2$

$\frac{dw}{dx} = 2x$

$dw = 2x\,dx$

$\frac{1}{2} dw = x\,dx$

$x \arctan x - \int \frac{1}{2} \frac{1}{w} dw$

$= x \arctan x - \frac{1}{2} \ln |w| + C$

$= x \arctan x - \frac{1}{2} \ln |1+x^2| + C$

$= \boxed{x \arctan - \frac{1}{2} \ln (1+x^2) + C}$

ex) $\int \arcsin x \, dx$

Recall:

$$\frac{d}{dx}\{\arcsin x\} = \frac{1}{\sqrt{1-x^2}}$$

$u : \arcsin x \quad du : \frac{1}{\sqrt{1-x^2}}\,dx$

$dv : dx \quad v : x$

$\int u\,dv = uv - \int v\,du$

$$= x\arcsin x - \int \frac{x}{\underbrace{\sqrt{1-x^2}}} dx$$

$-\frac{1}{2}\,dw$

$w = 1-x^2$

$dw = -2x\,dx$

$-\frac{1}{2}\,dw = x\,dx$

$$= x\arcsin x - \int -\frac{1}{2}\frac{1}{\sqrt{w}}\,dw = x\arcsin x + \frac{1}{2}\int w^{-1/2}\,dw$$

$$= x\arcsin x + \frac{1}{2}\cdot(2)\,w^{1/2} + C$$

$$= x\arcsin x + \sqrt{w} + C$$

$$= \boxed{x\arcsin x + \sqrt{1-x^2} + C}$$

"Integrating Around in a Circle"

$\int u\,dv = uv - \int v\,du$

(ex) $\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$

$u: e^x$ $\quad du: e^x\,dx$

$dv: \cos x\,dx$ $\quad v: \sin x$

$u: e^x$ $\quad du: e^x\,dx$

$dv: \sin x\,dx$ $\quad v: -\cos x$

$$= e^x \sin x + \left[+ e^x \cos x + \int - e^x \cos x\,dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$$

$+\int e^x \cos x\,dx$ to both sides

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$

$$\boxed{\int e^x \cos x\,dx = \frac{1}{2}\left[e^x \sin x + e^x \cos x\right] + C}$$

Integration by Parts:

$$\int u\,dv = uv - \int v\,du$$

(ex) $\boxed{\int e^x \sin x\,dx} =$

$u : e^x$ $\quad$ $du = e^x dx$

$du : \sin x\,dx$ $\quad$ $v = -\cos x$

$$-e^x \cos x - \int -e^x \cos x\,dx$$

$$= -e^x \cos x + \int e^x \cos x\,dx$$

$u : e^x$ $\quad$ $du = e^x dx$

$dv : \cos x\,dx$ $\quad$ $v = \sin x$

$$= \boxed{-e^x \cos x + e^x \sin x - \int e^x \sin x\,dx}$$

$+\int e^x \sin x\,dx$ to both sides

$$2\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x + C$$

$$\boxed{\int e^x \sin x\,dx = \frac{1}{2}\left[-e^x \cos x + e^x \sin x\right] + C}$$

(ex) $\int_0^{10} xe^x\,dx = xe^x\Big|_0^{10} - \int_0^{10} e^x\,dx$

$u: x \qquad du: 1\,dx$

$dv: e^x dx \qquad v: e^x$

$= (10e^{10} - 0) - \int_0^{10} e^x\,dx$

$= 10e^{10} - [e^{10} - e^0]$

$= 10e^{10} - e^{10} + e^0$

$= \boxed{9e^{10} + 1}$ Area under curve

(ex) $\int_0^{2\pi} x^2 \sin x\,dx = -x^2\cos x\Big|_0^{2\pi} - \int_0^{2\pi} -2x\cos x\,dx$

$u: x^2 \qquad du: 2x\,dx$

$dv: \sin x\,dx \qquad v: -\cos x$

$= -(2\pi)^2\cos(2\pi) - (-0)\cos 0 + \int_0^{2\pi} 2x\cos x\,dx$

$= -4\pi^2 + \int_0^{2\pi} 2x\cos x\,dx$

$u: 2x \qquad du = 2\,dx$

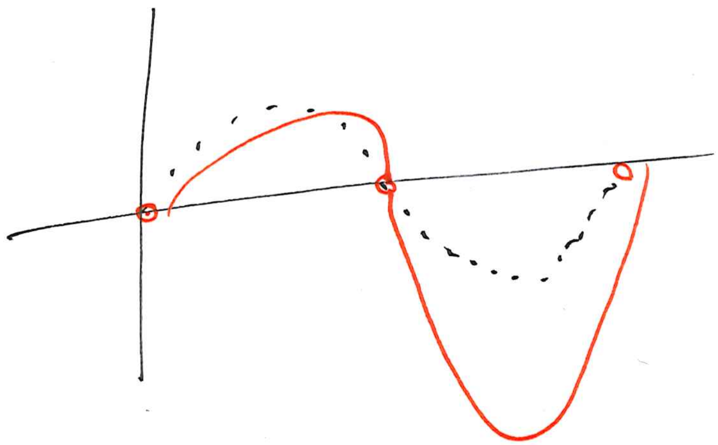
$dv: \cos x\,dx \qquad v: \sin x$

$= -4\pi^2 + 2x\sin x\Big|_0^{2\pi} - \int_0^{2\pi} 2\sin x\,dx$

$= -4\pi^2 + (4\pi\cancel{\sin(2\pi)}^{0} - 0) - \int_0^{2\pi} 2\sin x\,dx$

$= -4\pi^2 - 2\left[-\cos x\Big|_0^{2\pi}\right] = -4\pi^2 - 2(-1-(-1)) = \boxed{-4\pi^2}$

$x^2 \sin x$:



net area should be negative

$\boxed{\text{Trig Identities}}$

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$\cos^2 x = \frac{1 + \cos 2x}{2}$

Ch 7.3 : Trig Integrals

Integrating functions of the form $\sin^a x \cos^b x$ and $\sec^a x \tan^b x$

Recall: (1) $\sin^2 x + \cos^2 x = 1$

(2) $\tan^2 x + 1 = \sec^2 x$

(3) $\sin^2 x = \dfrac{1-\cos(2x)}{2}$

(4) $\cos^2 x = \dfrac{1+\cos(2x)}{2}$

Remark: If you forget (2), you can get it from (1):

$$\frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

(ex) $\int \underbrace{\sin^{10} x}_{u^{10}} \underbrace{\cos x\, dx}_{du} = \int u^{10}\, du = \frac{1}{11} u^{11} + C = \boxed{\frac{1}{11}\sin^{11} x + C}$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\, dx$

ex $\int \sin^{10}x \cdot \cos^5 x\, dx = \int \underbrace{\sin^{10}x}_{u^{10}} \cdot \underbrace{\cos^4 x}_{?} \cdot \underbrace{\cos x\, dx}_{du}$

need to change to sines

Idea: $u = \sin x$
$du = \cos x\, dx$

Identity: $\sin^2 x + \cos^2 x = 1$

so $\cos^2 x = 1 - \sin^2 x$

$\cos^4 x = (\cos^2 x)^2 = (1-\sin^2 x)^2$

$$= \int \underbrace{\sin^{10}x}_{u^{10}} \cdot \underbrace{(1-\sin^2 x)^2}_{(1-u^2)^2} \underbrace{\cos x\, dx}_{du} = \int u^{10} (1-u^2)^2 du$$

$$= \int u^{10}(1-2u^2+u^4)du = \int (u^{10} - 2u^{12} + u^{14})\, du = \frac{1}{11}u^{11} - 2\frac{1}{13}u^{13} + \frac{1}{15}u^{15} + C$$

$$= \boxed{\frac{1}{11}\sin^{11}x - \frac{2}{13}\sin^{13}x + \frac{1}{15}\sin^{15}x + C}$$

(ex) $\int \sin^5 x \cos^4 x \, dx$

Idea: $\int \sin^5 x \cdot (\cos^2 x)^2 dx$

$= \int \sin^5 x (1-\sin^2 x)^2 dx$

only sine: $u = \sin x$
$du = \cos x dx \leftarrow ??$

Another idea:

$\int \sin^4 x \cos^4 x \sin x dx$

$= \int (\sin^2 x)^2 \cos^4 x \sin x dx$

$= \int (1-\cos^2 x)^2 \cos^4 x \underbrace{\sin x dx}_{-du}$

$u = \cos x$
$du = -\sin x dx$
$-du = \sin x dx$

$-\int (1-u^2)^2 u^4 du = -\int (1-2u^2+u^4)u^4 du = -\int u^4 - 2u^6 + u^8 du$

$= -\left[\frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C\right] = \frac{-1}{5}u^5 + \frac{2}{7}u^7 - \frac{1}{9}u^9 + C$

$= \boxed{\frac{-1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + C}$

General Idea:

$$\int \sin^a x \cos^b x \, dx :$$

If a power is odd, reserve one of them as dx

So, u: use other

ex $$\int \sin^{17} x \cos^{16} x \, dx = \int \underbrace{\sin^{16} x}_{\text{convert to cosine}} \cos^{16} x \underbrace{\sin x \, dx}_{-du}$$

so: $u = \cos x$

ex $\displaystyle \int \sin^{2.71} x \cdot \cos^3 x \, dx = \int \sin^{2.71} x \cdot \underbrace{\cos^2 x} \cdot \underbrace{\cos x \, dx}_{du}$

$u : \sin x$

$$= \int \sin^{2.71} x \cdot (1 - \sin^2 x) \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int u^{2.71} (1 - u^2) \, du = \int \left(u^{2.71} - u^{4.71} \right) du$$

$$= \frac{u^{3.71}}{3.71} - \frac{u^{5.71}}{5.71} + C$$

$$= \boxed{\frac{(\sin x)^{3.71}}{3.71} - \frac{(\sin x)^{5.71}}{5.71} + C}$$

ex $\displaystyle\int \sin^5 x\,dx = \int \sin^4 x \underbrace{\sin x\,dx}_{\text{"}du\text{"}}$

$u: \cos x$

$= \displaystyle\int (\sin^2 x)^2 \sin x\,dx = \int (1-\cos^2 x)^2 \sin x\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$= \displaystyle\int (1-u^2)^2 (-1)\,du$ $\boxed{\text{etc}}$

What if powers are even?

ex $\int \sin^2 x\,dx$

use half-angle formulas:

$\sin^2 x = \frac{1-\cos 2x}{2}$ $\qquad$ $\cos^2 x = \frac{1+\cos 2x}{2}$

$$\int \sin^2 x\,dx = \int \frac{1-\cos(2x)}{2}\,dx = \frac{1}{2}\int 1-\cos(2x)\,dx$$

$u = 2x$
$du = 2\,dx$
$\frac{1}{2}du = dx$

$$= \frac{1}{2}\int [1-\cos(u)]\,\frac{1}{2}\cdot du$$

$$= \frac{1}{4}\int 1-\cos u\,du = \frac{1}{4}[u-\sin u] + C$$

$$= \boxed{\frac{1}{4}[2x-\sin(2x)] + C}$$

ex $\int \sin^2 x \cos^2 x \, dx$

Recall:

$\sin^2 x = \dfrac{1-\cos 2x}{2}$

$\cos^2 x = \dfrac{1+\cos 2x}{2}$

$= \int \left(\dfrac{1-\cos 2x}{2}\right)\left(\dfrac{1+\cos 2x}{2}\right) dx$

$= \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)\, dx = \frac{1}{4}\int \left[1-\cos^2(2x)\right] dx$

$\cos^2(2x) = \dfrac{1+\cos(4x)}{2}$

$= \frac{1}{4}\int \left[1 - \dfrac{1+\cos(4x)}{2}\right] dx$

$= \frac{1}{4}\int \left(1-\frac{1}{2}-\frac{1}{2}\cos(4x)\right) dx$

$= \frac{1}{4}\int \left(\frac{1}{2}-\frac{1}{2}\cos(4x)\right) dx$

$= \boxed{\frac{1}{4}\left[\frac{1}{2}x - \frac{1}{8}\sin(4x)\right] + C}$

Idea:

$\frac{1}{2}\int \cos^2 u \, du = \frac{1}{2}\int (1-\sin^2 u)\, du$

last ex

$\cancel{w = \sin u}$

$\cancel{dw = \cos u \, du}$???

Products of Secants and Tangents

Products of Secants and Tangents

$$\int \tan x\,dx = \ln|\sec x| + C$$

The antiderivative of the tangent function is the natural log of the absolute value of the secant function, plus any constant

$$\int \sec x\,dx = \int \sec x\left(\frac{\sec x+\tan x}{\sec x+\tan x}\right)dx = \int \frac{\overbrace{\sec^2 x+\sec x\tan x\;dx}^{du}}{\underbrace{\sec x+\tan x}_{u}}$$

$u = \sec x + \tan x$

$\frac{du}{dx} = \sec x\tan x + \sec^2 x$

$du = [\sec^2 x + \sec x \tan x]\,dx$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \boxed{\ln|\sec x+\tan x| + C}$$ ← memorize

ex $\int \sec^2 x \tan x \, dx$

1 way: $u = \tan x$ $\quad \int u \, du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\tan^2 x + C}$
$du = \sec^2 x \, dx$

Reserve $\boxed{2}$ secants (not one)

secant: even power

Another way: $u = \sec x$
$du = \sec x \tan x \, dx$

$\int \sec x \cdot \underbrace{\sec x \tan x \, dx}_{du} = \int u \, du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\sec^2 x + C}$

Note: $\boxed{\frac{1}{2}\tan^2 x} + C = \frac{1}{2}(\sec^2 x - 1) + C = \boxed{\frac{1}{2}\sec^2 x - \frac{1}{2}} + C$

$= \frac{1}{2}\sec^2 x + C$ arbitrary constant

Reserve: $\sec x \tan x$ as dx
Odd power of tangent
$u = \sec x$

Last Time:

$\int \sec^m x \cdot \tan^n x \, dx$:

Even power of secant:
Reserve $\sec^2 x$ for du
Other secants → tangents
$u = \tan x$

Odd power of tangent:
Reserve $\sec x \tan x$ for du
Other tangents → secants
$u = \sec x$

Odd power of secant, even power of tangent: reduction formula (we'll skip this)

IDENTITY:
$\tan^2 x + 1 = \sec^2 x$
so
$\tan^2 x = \sec^2 x - 1$

ex $\int \sec^3 x \tan^3 x \, dx = \int \sec^2 x \cdot \underline{\tan^2 x} \cdot \sec x \tan x \, dx$

$= \int \sec^2 x \, \underline{(\sec^2 x - 1)} \cdot \underbrace{\sec x \tan x \, dx}_{du} = \int u^2 (u^2 - 1) \, du$

$u = \sec x$
$du = \sec x \tan x \, dx$

$= \int (u^4 - u^2) \, du = \frac{u^5}{5} - \frac{u^3}{3} + C = \boxed{\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C}$

→ Able to do suggested problems through 7.3

Ch. 7.4 Trig Substitution

Motivation:

$$\int_3^7 \frac{1}{\sqrt{x^2+2x+1}}\,dx = \int_3^7 \frac{1}{\cancel{\sqrt{(x+1)^2}}}\,dx$$

$$= \int_3^7 \frac{1}{x+1}\,dx = \ln|x+1|\Big|_3^7$$

$$= \ln 8 - \ln 4 = \ln(8/4) = \boxed{\ln 2}$$

Nice thing:

$\cancel{\sqrt{(\ )^2}}$

get rid of $\sqrt{\ }$

Very similar integrand

function we're integrating

Goal:

$\sqrt{x^2+1} = \cancel{\sqrt{(\quad)^{\cancel{2}}}}$

cancel $\sqrt{\ }$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx$$

Recall: $(\tan\theta)^2+1 = (\sec\theta)^2$

So, if $x=\tan\theta$: $x^2+1 = \tan^2\theta+1 = (\sec\theta)^2$

then: $\sqrt{x^2+1} = \cancel{\sqrt{(\sec\theta)^{\cancel{2}}}} = \sec\theta$

Sub: $x=\tan\theta$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sec\theta}\cdot \sec^2\theta\, d\theta = \int \sec\theta\, d\theta$$

(last time - memorize)

$$= \ln|\underbrace{\sec\theta}_{\sqrt{x^2+1}} + \underbrace{\tan\theta}_{x}| + C$$

$\theta \to x$

$$= \boxed{\ln|\sqrt{x^2+1} + x| + C}$$

Sub: $x = \tan\theta$

what is $\sec\theta$?

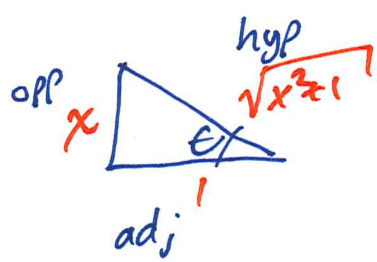
$\boxed{2 \text{ WAYS}}$

- 1^{st} Calc: $\sqrt{x^2+1} = \sec\theta$

- Draw a triangle:

$x = \tan\theta$

$\frac{x}{1} = \frac{opp}{adj}$



Then: $\sec\theta = \frac{hyp}{adj} = \frac{\sqrt{x^2+1}}{1}$

So $\sec\theta = \sqrt{x^2+1}$

Idea: $\int \frac{1}{\sqrt{1+x^2}}\,dx = \int (1+x^2)^{-1/2}\,dx = \dots ?$

Ex $\int x\sqrt{9-x^2}\,dx$

Note: Easier to solve using $u = 9-x^2$
To practice the method, we'll use trig sub

Form: $\sqrt{\text{(quadratic)}} \xrightarrow[\text{goal}]{} \cancel{\sqrt{(\quad)^{2}}}$ get rid of $\sqrt{\ }$

Choose substitution

have $9 - x^2$

const − fcn

$1 - \sin^2\theta = \cos^2\theta$

want $9 - 9\sin^2\theta = 9\cos^2\theta$

Closest identity

fix const

Need: $x^2 = 9\sin^2\theta$

Use: $\boxed{x = 3\sin\theta}$

Identities:

$1-\sin^2\theta = \cos^2\theta$

$1+\tan^2\theta = \sec^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

const − fcn

const + fcn

fcn − const

Check that it's a good idea by looking ahead

($\sqrt{\ } \to$ cancel !)

$$9-x^2 = 9-(3\sin\theta)^2$$
$$= 9-9\sin^2\theta$$
$$= 9(1-\sin^2\theta)$$
$$= 9\cos^2\theta$$

so: $\sqrt{9-x^2} = \cancel{\sqrt{9\cos^2\theta}}$
$= 3\cos\theta$ $\quad$ $\sqrt{\ }$ went away – good substitution!

Do substitution: $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$

$$\int x\sqrt{9-x^2}\,dx = \int 3\sin\theta \cdot 3\cos\theta \cdot 3\cos\theta\, d\theta = \int 27\cdot\cos^2\theta\sin\theta\, d\theta$$

$u = \cos\theta$
$-du = \sin\theta\, d\theta$

Evaluate:

$$-27\int u^2\,du = -27\cdot\frac{1}{3}u^3 + C$$

Get original variable back

$$-9u^3 + C \underset{\substack{\uparrow \\ u = \cos\theta}}{=} -9 \cdot \cos^3\theta + C = -9 \cdot \left(\frac{1}{3}\sqrt{9-x^2}\right)^3 + C$$

Already did:

$\sqrt{9-x^2} = 3\cos\theta$

so $\frac{1}{3}\sqrt{9-x^2} = \cos\theta$

$$= \frac{-9}{27}(9-x^2)^{3/2} + C = \boxed{\frac{-1}{3}(9-x^2)^{3/2} + C}$$

$$\sqrt{A}^3 = (A^{1/2})^3 = A^{3/2}$$

ex $\int \frac{1}{(x^2-16)^{3/2}}\,dx = \int \frac{1}{\sqrt{x^2-16}^{\,3}}\,dx$

want $\sqrt{\ }$ go away

$1-\sin^2\theta = \cos^2\theta$ const - fcn

$1+\tan^2\theta = \sec^2\theta$ const + fcn

$\sec^2\theta - 1 = \tan^2\theta$ fcn - const ← closest

have: x^2-16

fcn - const

$\sec^2\theta - 1 = \tan^2\theta$

fix constant

want: $16\sec^2\theta - 16 = 16\tan^2\theta$

$x^2 = 16\sec^2\theta$

use $\boxed{x = 4\sec\theta}$ as substitution

Check that $x = 4\sec\theta$ really gets rid of $\sqrt{\ }$

$$x^2 - 16 = (4\sec\theta)^2 - 16$$
$$= 16\sec^2\theta - 16$$
$$= 16(\sec^2\theta - 1)$$
$$= 16 \cdot \tan^2\theta$$

So: $\sqrt{x^2-16} \cancel{=} \sqrt{16 \cdot \tan^2\theta}$
$$= 4\tan\theta$$

$\sqrt{\ }$ went away – good substitution!

Do substitution: $x = 4\sec\theta$

$dx = 4\sec\theta \tan\theta \, d\theta$

$$\int \frac{1}{\sqrt{x^2-16}^{\,3}}\,dx = \int \frac{1}{(4\tan\theta)^3} 4\sec\theta\tan\theta\,d\theta = \frac{1}{16}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

Evaluate $\frac{1}{16}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{16}\int \frac{1}{\cos\theta}\cdot\left(\frac{\cos\theta}{\sin\theta}\right)^2 d\theta$

$$= \frac{1}{16}\int \frac{\overbrace{\cos\theta\;d\theta}^{du}}{\sin^2\theta} = \frac{1}{16}\int \frac{1}{u^2}\,du = \int u^{-2}\,du$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= -\frac{1}{16}u^{-1} + C$$

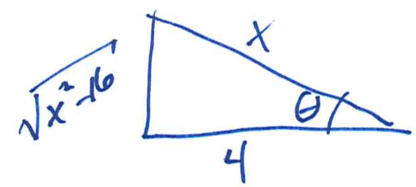
Get original variable back

$$= \frac{-1}{16}\cdot(\sin\theta)^{-1} + C = \frac{-1}{16\sin\theta} + C$$

$$= \boxed{\frac{-1}{16}\cdot\frac{x}{\sqrt{x^2-16}} + C}$$

Used: $x = 4\sec\theta$

$\frac{hyp}{adj} = \sec\theta = \frac{x}{4}$



$\sin\theta = \frac{opp}{hyp} = \frac{\sqrt{x^2-16}}{x}$

(ex) $\int \frac{\sqrt{4x^2-1}}{x}\,dx$

Identities:

$1-\sin^2\theta = \cos^2\theta$

$1+\tan^2\theta = \sec^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$4x^2 - 1$

fcn - const

$\sec^2\theta - 1 = \tan^2\theta$

Need: $4x^2 = \sec^2\theta$

$\boxed{2x = \sec\theta}$

Check substitution:

$4x^2-1 = (2x)^2 - 1 = (\sec\theta)^2 - 1 = \tan^2\theta$

So: $\sqrt{4x^2-1} = \cancel{\sqrt{\tan^2\theta}} = \tan\theta$

$\sqrt{\ }$ cancelled: good sub!

Do subst:

$2x = \sec\theta$

$x = 1/2 \sec\theta$

$dx = \frac{1}{2}\sec\theta\tan\theta\, d\theta$

$$\int \frac{\sqrt{4x^2-1}}{x}\, dx$$

$$= \int \frac{\tan\theta}{\cancel{\frac{1}{2}\sec\theta}} \cancel{\tfrac{1}{2}\sec\theta} \tan\theta\, d\theta = \int \tan^2\theta\, d\theta$$

$$= \int (\sec^2\theta - 1)\, d\theta = \tan\theta - \theta + C$$

used:

$x = \frac{1}{2}\sec\theta$

$2x = \sec\theta$

$\operatorname{arcsec}(2x) = \theta$

$\tan\theta = \sqrt{4x^2-1}$

Need to get x back:

$$= \boxed{\sqrt{4x^2-1} - \operatorname{arcsec}(2x) + C}$$

Completing the Square

ex $\int \frac{1}{\sqrt{3-x^2+2x}}\,dx$

3 pieces

$\downarrow$ complete $\square$

2 pieces

$\downarrow$ trig

1 piece $\cancel{\sqrt{(\ )^2}}$

$3-x^2+2x =$

$-[\underbrace{x^2-2x}-3]$

$= -[\underbrace{x^2-2x+1}\underbrace{-1-3}]$

$= -[(x-1)^2-4]$

$= 4-(x-1)^2$

Recall:

$(x+a)^2 = \underbrace{x^2+2ax}+a^2$

$a=-1$

$(x-1)^2 = x^2-2x+1$

Choose sub:

const − fcn

$$1 - \sin^2\theta = \cos^2\theta$$

have: $4 - (x-1)^2$

$$1 - \sin^2\theta = \cos^2\theta$$

match constants $4 - 4\sin^2\theta = 4\cos^2\theta$

Need: $(x-1)^2 = 4\sin^2\theta$

$$\boxed{x - 1 = 2\sin\theta}$$

Also can say

$$\boxed{x = 1 + 2\sin\theta}$$

Check our sub:

$$3 - x^2 + 2x = 4 - (x-1)^2$$
$$= 4 - (2\sin\theta)^2$$
$$= 4 - 4\sin^2\theta$$
$$= 4(1 - \sin^2\theta)$$
$$= 4\cos^2\theta$$

$x = 1 + 2\sin\theta$
$dx = 2\cos\theta\, d\theta$

Then: $\sqrt{3 - x^2 + 2x} = \sqrt{4\cos^2\theta} = 2\cos\theta$ $\sqrt{\ }$ gone!

$$\int \frac{1}{\sqrt{3 - x^2 + 2x}}\, dx = \int \frac{1}{2\cos\theta} \cdot 2\cos\theta\, d\theta = \int 1\, d\theta = \theta + C$$

$$= \boxed{\arcsin\left(\frac{x-1}{2}\right) + C}$$

$x - 1 = 2\sin\theta$

$\frac{x-1}{2} = \sin\theta$

$\theta = \arcsin\left(\frac{x-1}{2}\right)$

→ Suggested Prb: $\boxed{§7.4}$

Feb 28

Ch 7.5 : Partial Fractions

Motivation: Fact: $\dfrac{1}{x+1} - \dfrac{1}{2x-1} = \dfrac{x-2}{(x+1)(2x-1)}$

$\int \dfrac{1}{x+1} - \dfrac{1}{2x-1}\, dx$: easy enough

$$\underbrace{\int \frac{1}{x+1}\, dx}_{u = x+1} - \underbrace{\int \frac{1}{2x-1}\, dx}_{u = 2x-1} \quad \text{etc}$$

$\int \dfrac{x-2}{(x+1)(2x-1)}\, dx$: pretty tough

Method of Partial Fractions:

re-write a rational function as a sum of rational functions that are easy to integrate.

$\frac{\text{polynomial}}{\text{polynomial}}$

(just algebra!)

1st case: Denominator – Repeated Linear Factors

$$\frac{\text{numerator}}{(ax+r)^n} = \frac{C_1}{ax+r} + \frac{C_2}{(ax+r)^2} + \frac{C_3}{(ax+r)^3} + \dots + \frac{C_n}{(ax+r)^n}$$

numerator: polynomial, degree $< n$

a, r const

n natural

ex $\displaystyle\int \frac{6x+7}{\underbrace{4x^2+20x+25}_{\text{rational}}}\,dx$

denominator: $(2x+5)^2$

$$\frac{6x+7}{(2x+5)^2} = \underbrace{\frac{C}{(2x+5)} + \frac{D}{(2x+5)^2}}_{\text{easier to } \int}$$

Find C, D

$$= \frac{C(2x+5)+D}{(2x+5)^2}$$

common denominator: $(2x+5)^2$

$$\underbrace{6}x+\underbrace{7} = C(2x+5)+D = \underbrace{(2C)}x + \underbrace{(5C+D)}$$

$6 = 2C \longrightarrow \boxed{C=3}$

and $7 = 5C+D$

$7 = 5\cdot 3 + D$

$= 15 + D$

so $\boxed{D=-8}$

$$\int \frac{6x+7}{(2x+5)^2}\,dx = \int \frac{3}{2x+5} + \frac{-8}{(2x+5)^2}\,dx = \int \left[\frac{3}{u} - \frac{8}{u^2}\right]\cdot\frac{1}{2}\cdot du$$

$u = 2x+5$

$du = 2dx$

$\frac{1}{2}du = dx$

$$= \frac{1}{2}\int \frac{3}{u} - 8u^{-2}\,du = \frac{1}{2}\left[3\ln|u| + 8u^{-1}\right] + C$$

$$= \boxed{\frac{1}{2}\left[3\ln|2x+5| + \frac{8}{2x+5}\right] + C}$$

(ex) $$\int \frac{x^2+6x+10}{(x+3)^3}\,dx$$

$$\frac{x^2+6x+10}{(x+3)^3} = \frac{C}{(x+3)} + \frac{D}{(x+3)^2} + \frac{E}{(x+3)^3}$$

Find C, D, E

$$= \frac{C(x+3)^2 + D(x+3) + E}{(x+3)^3}$$

Common Denominator $(x+3)^3$

$$x^2+6x+10 = C(x+3)^2 + D(x+3) + E$$

if $x=-3$:

$$9-18+10 = 0+0+E$$

$\boxed{1=E}$

$\boxed{C=1}$

$$x^2+6x+10 = C(x^2+6x+9) + D(x+3) + 1$$

$$= x^2(C) + x(6C+D) + (9C+3D+1)$$

$$= x^2 + x(6+D) + (10+3D)$$

$6+D=6$

$\boxed{D=0}$

$$\int \frac{x^2+6x+10}{(x+3)^3}\,dx = \int \frac{1}{x+3} + \frac{1}{(x+3)^3}\,dx = \text{etc.}$$

Case 2: Denom has distinct linear factors

all differnt
no 2 same

Rule: $$\frac{\text{num}}{(a_1x+r_1)(a_2x+r_2)\cdots(a_nx+r_n)} = \frac{A}{(a_1x+r_1)} + \frac{B}{(a_2x+r_2)} + \cdots + \frac{C}{(a_nx+r_n)}$$

a_i, r_i const

num: polynomial, degree $< n$

a_ix+r_i all differnt

ex $\int \frac{7x+13}{2x^2+x-10}\,dx$

$$\frac{7x+13}{(2x+5)(x-2)} = \frac{A}{2x+5} + \frac{B}{x-2}$$

Find A, B

$$= \frac{A(x-2) + B(2x+5)}{(2x+5)(x-2)}$$

common denom

$$\underbrace{7}x + \underbrace{13} = Ax - 2A + 2Bx + 5B$$

$$= x\underbrace{(A+2B)} + \underbrace{(-2A+5B)}$$

$7 = A + 2B \rightarrow A = 7 - 2B$

$13 = -2A + 5B$

$13 = -2(7-2B) + 5B$

$13 = -14 + 4B + 5B$

$27 = 9B$

$\boxed{B=3}$

$A = 7 - 2(3)$

$\boxed{A=1}$

$$\int \frac{7x+13}{(2x+5)(x-2)}\,dx = \underbrace{\int \frac{1}{2x+5} + \frac{3}{x-2}\,dx}_{\text{easier}}$$

Case 3: Some distinct, some repeated linear factors in denom

ex $$\frac{4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \qquad \text{etc.}$$

Possible Complication: deg of num $\geq$ deg of denom

In this case: DIVIDE

ex: $$\frac{13}{3} = \frac{12+1}{3} = \frac{4\cdot 3+1}{3} = \frac{4\cdot\cancel{3}}{\cancel{3}} + \frac{1}{3} = 4 + \frac{1}{3}$$

pulled out biggest multiple of denom from num

separate fraction + cancel

ex: $$\frac{x^2+4x-7}{x^2-3x+2} = \frac{x^2-3x+2 \;\; +7x-9}{x^2-3x+2} = \frac{x^2-3x+2}{x^2-3x+2} + \frac{7x-9}{x^2-3x+2}$$

deg of num = deg of denom so partial fractions won't work (yet)

$$= 1 + \underbrace{\frac{7x-9}{x^2-3x+2}} = 1 + \frac{7x-9}{(x-1)(x-2)}$$ $\boxed{\text{etc...}}$

deg of num < deg of denom: can do partial fractions (ex) $$\frac{x^3 - 3x^2 + 9x - 9}{x^2 - 3x + 2} = \frac{x^3 - 3x^2 + 2x \quad + 7x - 9}{x^2 - 3x + 2}$$

Can't do partial fractions yet - deg of num too big

$$= \frac{x^3 - 3x^2 + 2x}{x^2 - 3x + 2} + \frac{7x - 9}{x^2 - 3x + 2}$$

Note: $x(x^2 - 3x + 2) =$
$x^3 - 3x^2 + 2x$

$$= \frac{x\cancel{(x^2 - 3x + 2)}}{\cancel{x^2 - 3x + 2}} + \frac{7x - 9}{x^2 - 3x + 2}$$

$$= \underbrace{x + \frac{7x - 9}{x^2 - 3x + 2}}_{\text{now: can do partial fractions}}$$

ex $\frac{2x^3-5x^2+8x-7}{x^2-3x+2} = \frac{2x^3-6x^2+4x+x^2+4x-7}{x^2-3x+2}$

Note: $2x(x^2-3x+2)$
$= 2x^3-6x^2+4x$

$= \frac{2x\cancel{(x^2-3x+2)}}{\cancel{x^2-3x+2}} + \frac{x^2+4x-7}{x^2-3x+2}$

$= 2x + \frac{x^2+4x-7}{x^2-3x+2} = 2x + \frac{x^2-3x+2 \quad +7x-9}{x^2-3x+2}$

Note: x^2-3x+2

$= 2x + \frac{x^2-3x+2}{x^2-3x+2} + \frac{7x-9}{x^2-3x+2}$

$= 2x+1+\underbrace{\frac{7x-9}{x^2-3x+2}}_{\text{can do part frac}}$

All suggested HW through 7.5

Ch 7.7: Numerical Integration

Motivation: sometimes we can't (don't want to) find antiderivative

Recall: $\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

ex $\int e^{x^2}\,dx$

ex $\int \frac{1}{\ln x}\,dx$

ex $\int \sin(x^2)\,dx$

Absolute vs Relative Error

Absolute Error:

$|\text{exact} - \text{approx}|$

Relative Error:

$\dfrac{\text{abs error}}{|\text{actual}|}$

Case 1: 500g sack of flour mistakenly labeled 495g

$|500 - 495| = 5g$

$\frac{5}{500} = \frac{1}{100} = 1\%$

Case 2: 5g bottle of medicine mistakenly labeled 10g

$|5 - 10| = 5g$

$\frac{5}{5} = 1 = 100\%$

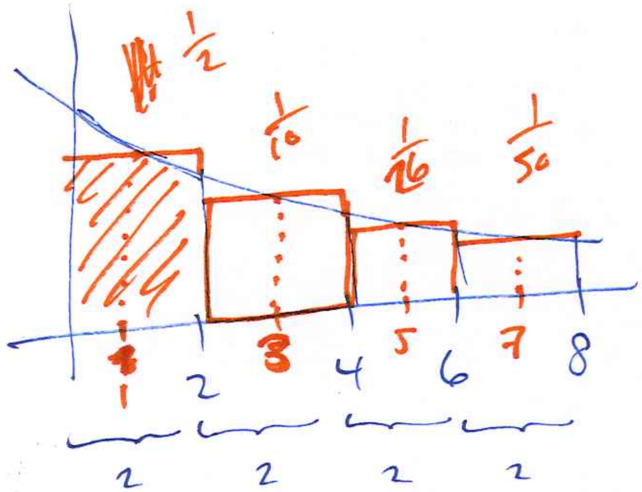
(ex) We already saw midpt Riemann Sums

If we take n intervals (~~not~~ limit)

"Midpt Approximation"

(ex) approx $\int_0^8 \frac{1}{1+x^2}\,dx$ using midpt approx, $n=4$

$\approx 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{10}\right) + 2\left(\frac{1}{26}\right) + 2\left(\frac{1}{50}\right)$

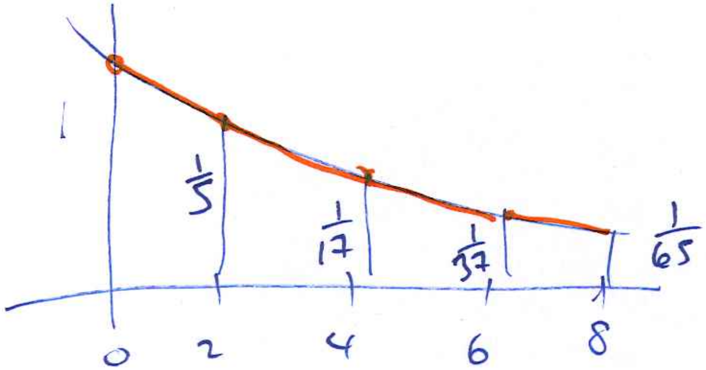


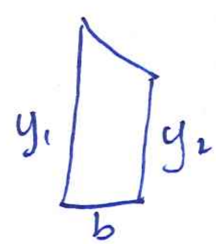
Approx fcn by constant line

(ex) $\int_0^8 \frac{1}{1+x^2}\,dx$

Approx fcn using lines

"Trapezoid Rule"





Area of trapezoid:

$\frac{1}{2} b(y_1 + y_2)$

$$\frac{1}{2}(2)\left(1+\frac{1}{5}\right)+\frac{1}{2}(2)\left(\frac{1}{5}+\frac{1}{17}\right)+\frac{1}{2}(2)\left(\frac{1}{17}+\frac{1}{37}\right)+\frac{1}{2}2\left(\frac{1}{37}+\frac{1}{65}\right)$$

General Form

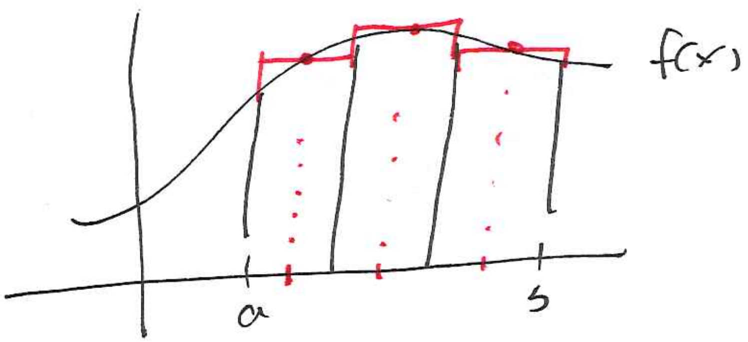
Trapezoid: $$\int_a^b f(x)\,dx \approx \Delta x\left(\frac{1}{2}f(x_0)+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+\frac{1}{2}f(x_n)\right)$$

where $\Delta x = \frac{b-a}{n}$, $x_k = a+k\Delta x$

Midpoint: $$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\Delta x$$

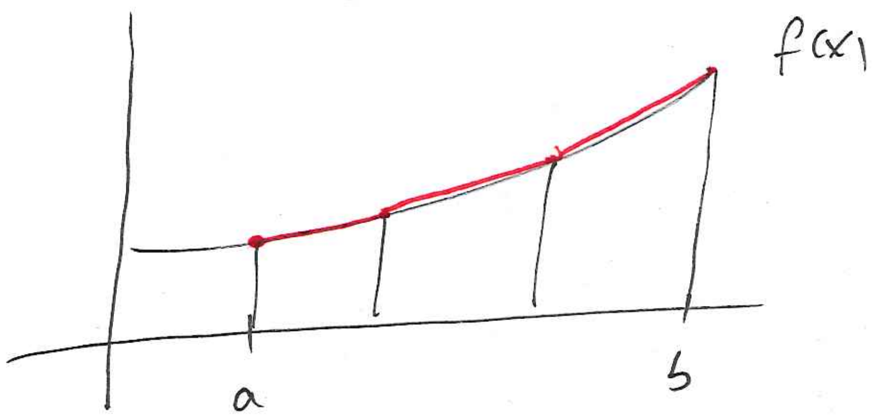
2 March 2017

Midpoint Rule (p. 559)



Approximating f(x) by a constant (in each interval)

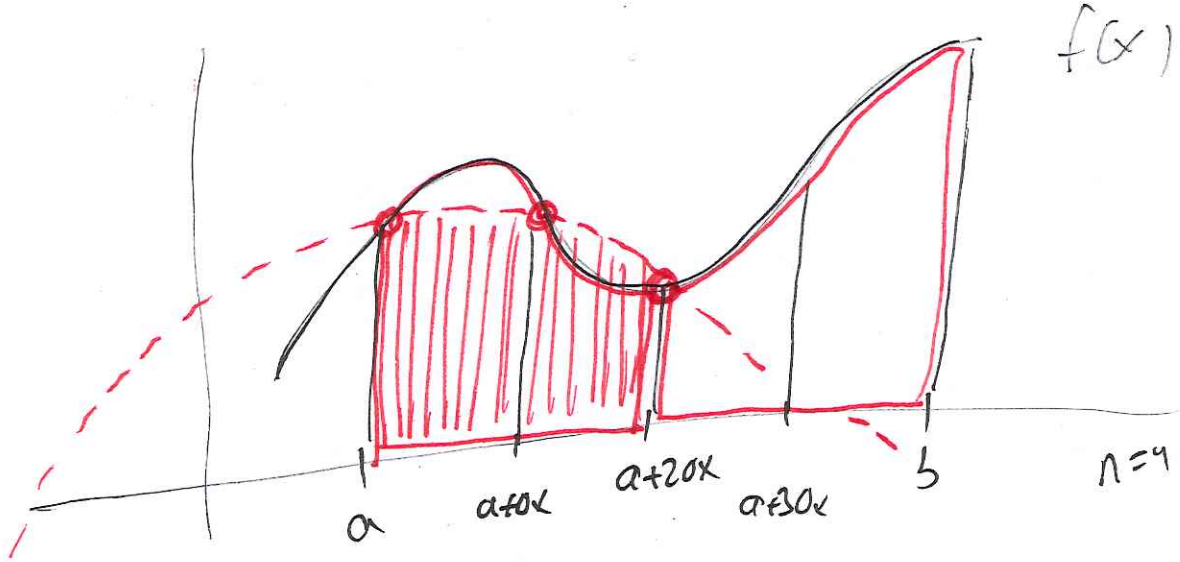
Trapezoid Rule (p. 560)



Approximating f(x) by a line (in each interval)

Simpson's Rule:

Approx f(x) by
a parabola
(in each interval)



Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 4f(x_{n-1}) + f(x_n) \Big]$$

$\Delta x = \frac{b-a}{n}$

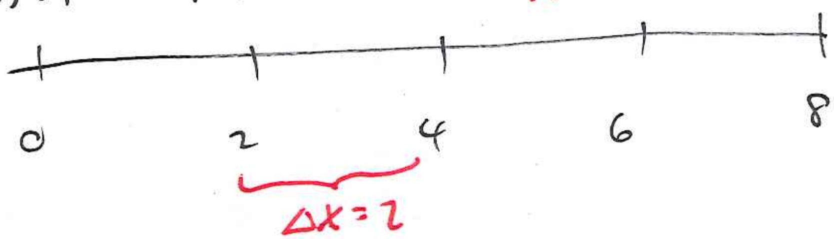
$\boxed{\text{only when } n \text{ even}}$

ex Use Simpson's Rule, $n=4$ intervals

$$\int_0^8 \frac{1}{1+x^2}\,dx \approx \frac{2}{3}\left[1+4\cdot\frac{1}{5}+2\cdot\frac{1}{17}+4\cdot\frac{1}{37}+\frac{1}{65}\right]$$

$f(0)=1 \quad f(2)=\frac{1}{5} \quad f(4)=\frac{1}{17} \quad f(6)=\frac{1}{37} \quad f(8)=\frac{1}{65}$

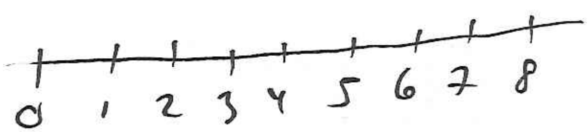
$\Delta x=\frac{b-a}{n}=\frac{8}{4}=2$



ex Use Simpson's Rule, 8 intervals

$$\int_0^8 e^{x^2}\,dx \approx \frac{1}{3}\left[e^0+4e^1+2e^4+4\cdot e^9+2e^{16}+4e^{25}+2e^{36}+4e^{49}+e^{64}\right]$$

$\Delta x=\frac{b-a}{n}=\frac{8-0}{8}=1$



Formulas for Error: formula sheet

Theorem 7.2, p565

ex Find error involved with approximating $\int_0^1 \sin(2x)\,dx$ using 10 intervals (all 3 methods)

$b = 1$ $\quad (b-a) = 1$

$a = 0$ $\quad \Delta x = \frac{b-a}{n} = \frac{1}{10}$

$n = 10$

$f(x) = \sin(2x)$

$f'(x) = 2\cos(2x)$

$\boxed{f''(x) = -4\sin(2x)}$ $\quad |-4\sin(2x)| \leq 4$

$f'''(x) = -8\cos(2x)$

$\boxed{f^{(4)}(x) = 16\sin 2x}$ $\quad |16\sin(2x)| \leq 16$

$4 = k$

$16 = K$

Using Midpoint: $|\text{Error}| \leq \frac{k(b-a)}{24}(\Delta x)^2 = \frac{4\cdot 1}{24}\left(\frac{1}{10}\right)^2 = \frac{1}{6}\cdot\frac{1}{100} = \boxed{\frac{1}{600}}$

Using Trapezoid: $|\text{Error}| \leq \frac{k(b-a)}{12}(\Delta x)^2 = \frac{4\cdot 1}{12}\left(\frac{1}{10}\right)^2 = \frac{1}{3}\cdot\frac{1}{100} = \boxed{\frac{1}{300}}$

Possible: (eg) error using MP: $\frac{1}{600}$

error using : $\frac{1}{1000}$

Error)
Using Simpson's Rule: $\le \dfrac{K(b-a)}{180}(\Delta x)^4$

$$= \frac{16(1)}{180}\left(\frac{1}{10}\right)^4 = \frac{1}{112{,}500}$$

(ex) What is the error involved with Simpson's Rule approximating $\int_1^2 \frac{1}{x}\,dx$ using 6 intervals?

$$E_S \le \frac{K(b-a)}{180}(\Delta x)^4 = \frac{24(2-1)}{180}\Big(\underbrace{\frac{2-1}{6}}_{\Delta x}\Big)^4$$

K : upper-bound on $|f^{(4)}(x)|$

$$= \frac{24}{180\cdot 6^4} \stackrel{?}{=} \frac{1}{38{,}880}$$

$f(x) = \frac{1}{x} = x^{-1}$
$f'(x) = -x^{-2}$
$f''(x) = 2x^{-3}$
$f'''(x) = -6x^{-4}$
$f^{(4)}(x) = 24x^{-5} = \frac{24}{x^5}$

$\left|\frac{24}{x^5}\right| \le \frac{24}{1} = 24$

Use $K = 24$

Note: $\int_1^2 \frac{1}{x}\,dx = \ln 2 - \ln 1 = \boxed{\ln 2}$
exact

Suppose you want to approx $\ln 2 = \int_1^2 \frac{1}{x}\,dx$ using midpoint rule, your error should be at most 10^{-4}.

How many intervals do you need?

$$E_M \leq \frac{k(b-a)}{24}(\Delta x)^2 \underset{\text{WANT}}{\leq} 10^{-4}$$

$f(x) = \frac{1}{x}$

$f'(x) = \frac{-1}{x^2}$

$\boxed{f''(x) = \frac{2}{x^3}}$ → when x is between 1 & 2,

$|f''(x)| = \left|\frac{2}{x^3}\right| \leq \frac{2}{1} = 2$

Use: $\boxed{k=2}$

$$\frac{k(b-a)}{24}(\Delta x)^2 \overset{want}{\leq} 10^{-4}$$

$$\frac{2(2-1)}{24}\left(\frac{\cancel{b-a}^{\,1}}{n}\right)^2 \leq \frac{1}{10^4}$$

$$\frac{1}{12}\cdot\frac{1}{n^2} \leq \frac{1}{10^4}$$

$$12n^2 \geq 10^4$$

$$n^2 \geq \frac{10^4}{12}$$

$$n \geq \sqrt{\frac{10^4}{12}} = \frac{100}{\sqrt{12}} \approx 28.8$$

use $\boxed{29}$ intervals.

→ Suggested Probs §7.7

§7.8 Improper Integrals

Two ways for an integral to be improper:

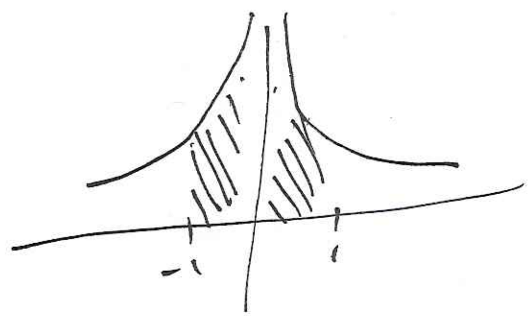
- infinite interval of integration
- integrand ($f(x)$) not bounded on region of integration

example: $\int_1^\infty \frac{1}{x^2}\,dx$

improper

$\int_0^1 \frac{1}{x^2}\,dx$, $\int_{-1}^1 \frac{1}{x^2}\,dx$

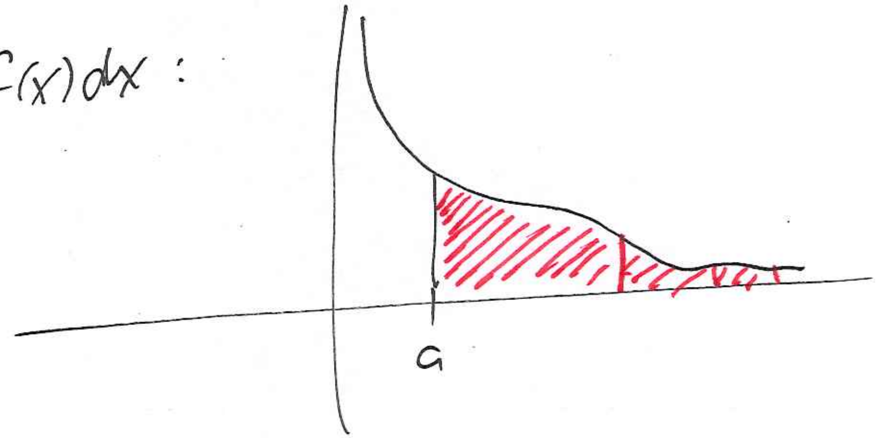
improper



Infinite Interval

We use a limit

$\int_a^\infty f(x)\,dx$:



ex $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$

$= \lim_{b\to\infty}\left[\frac{-1}{x}\Big|_1^b \right]$

$= \lim_{b\to\infty}\left[\frac{-1}{b} - \frac{-1}{1}\right] = \lim_{b\to\infty}\left[\frac{-1}{b} + 1\right] = \boxed{1}$

ex $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx \neq \lim_{a\to\infty} \int_{-a}^{a} \frac{1}{1+x^2}\,dx$

$\parallel$

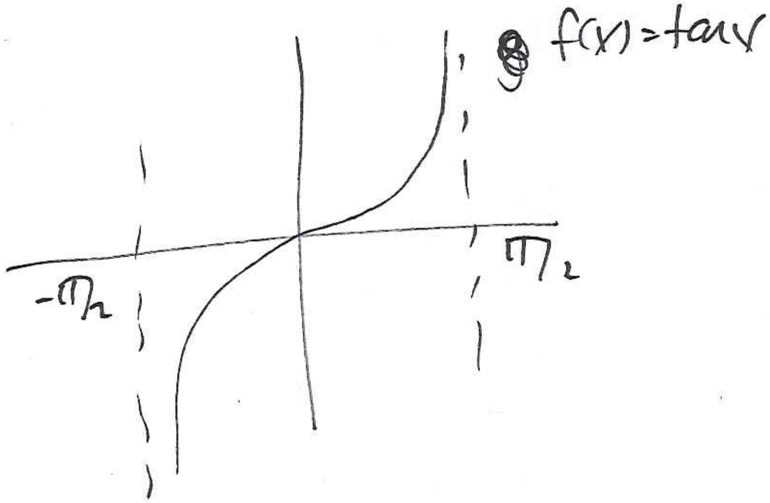
$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty}\left[\int_a^0 \frac{1}{1+x^2}\,dx\right] + \lim_{b\to\infty}\left[\int_0^b \frac{1}{1+x^2}\,dx\right)$

$= \lim_{a\to-\infty}\left[\overset{0}{\cancel{\arctan 0}} - \arctan a\right] + \lim_{b\to\infty}\left[\arctan b - \overset{0}{\cancel{\arctan 0}}\right]$

$= -\left[-\pi/2\right] + \pi/2 = \pi/2 + \pi/2 = \boxed{\pi}$

Aside: $\arctan(x) = y$

means: $\tan(y) = x$



As $y \to \pi/2$, $\tan(y) \to \infty$ } "arctan ∞" "=" "π/2"

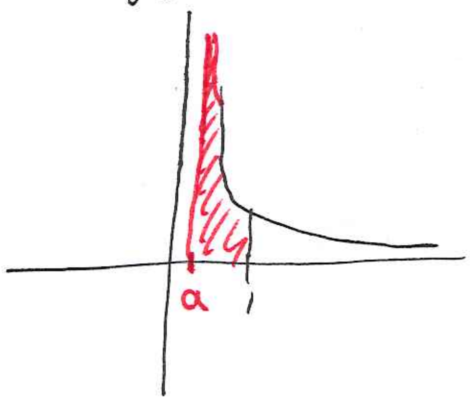
$$\lim_{x \to \infty} \arctan x = \pi/2$$

$$\lim_{x \to \pi/2^-} \tan x = \infty$$

$$\lim_{x \to -\infty} \arctan x = -\pi/2$$

Unbounded Function

(ex) $\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\left[\int_a^1 \frac{1}{x^2}\,dx\right] = \lim_{a\to 0^+}\left[\frac{-1}{x}\Big|_a^1\right]$



$= \lim_{a\to 0^+}\left[\frac{-1}{1} - \frac{-1}{a}\right]$

$= \lim_{a\to 0^+}\left[-1 + \frac{1}{a}\right] = \infty$

We say this integral <u>diverges</u>
(limit doesn't exist!)

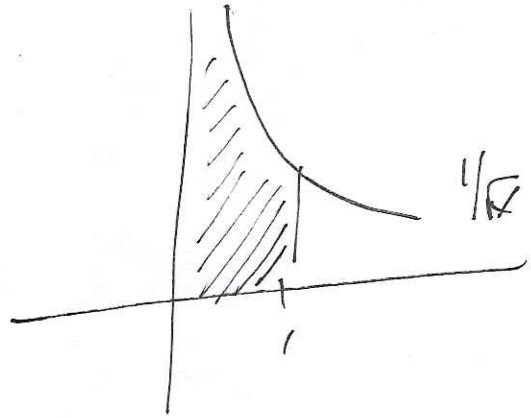
By the way: If limit gives a finite number, we say the integral <u>converges</u>.

ex $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\left[\int_a^1 x^{-1/2}\,dx\right]$

$= \lim_{a\to 0^+}\left[\; 2x^{1/2}\Big|_a^1 \right]$

$= \lim_{a\to 0^+}\left[2\sqrt{1} - 2\sqrt{a}\right]$

$= \boxed{2}$



Improper Integrals

ex $$\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+}\left[\int_a^1 \frac{1}{x}\,dx\right] = \lim_{a\to 0^+}\left[\cancel{\ln 1}^{\,0} - \ln a\right] = \infty$$

DIVERGES

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\int_1^b \frac{1}{x}\,dx\right] = \lim_{b\to\infty}\left[\ln b - \cancel{\ln 1}^{\,0}\right] = \infty$$

DIVERGES

ex $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[\int_1^b x^{-2}\,dx\right] = \lim_{b\to\infty}\left[-x^{-1}\Big|_1^b\right]$$

CONVERGES $$= \lim_{b\to\infty}\left[-\frac{1}{b} - \frac{-1}{1}\right] = \lim_{b\to\infty}\left[1 - \cancel{\frac{1}{b}}^{\,0}\right] = 1$$

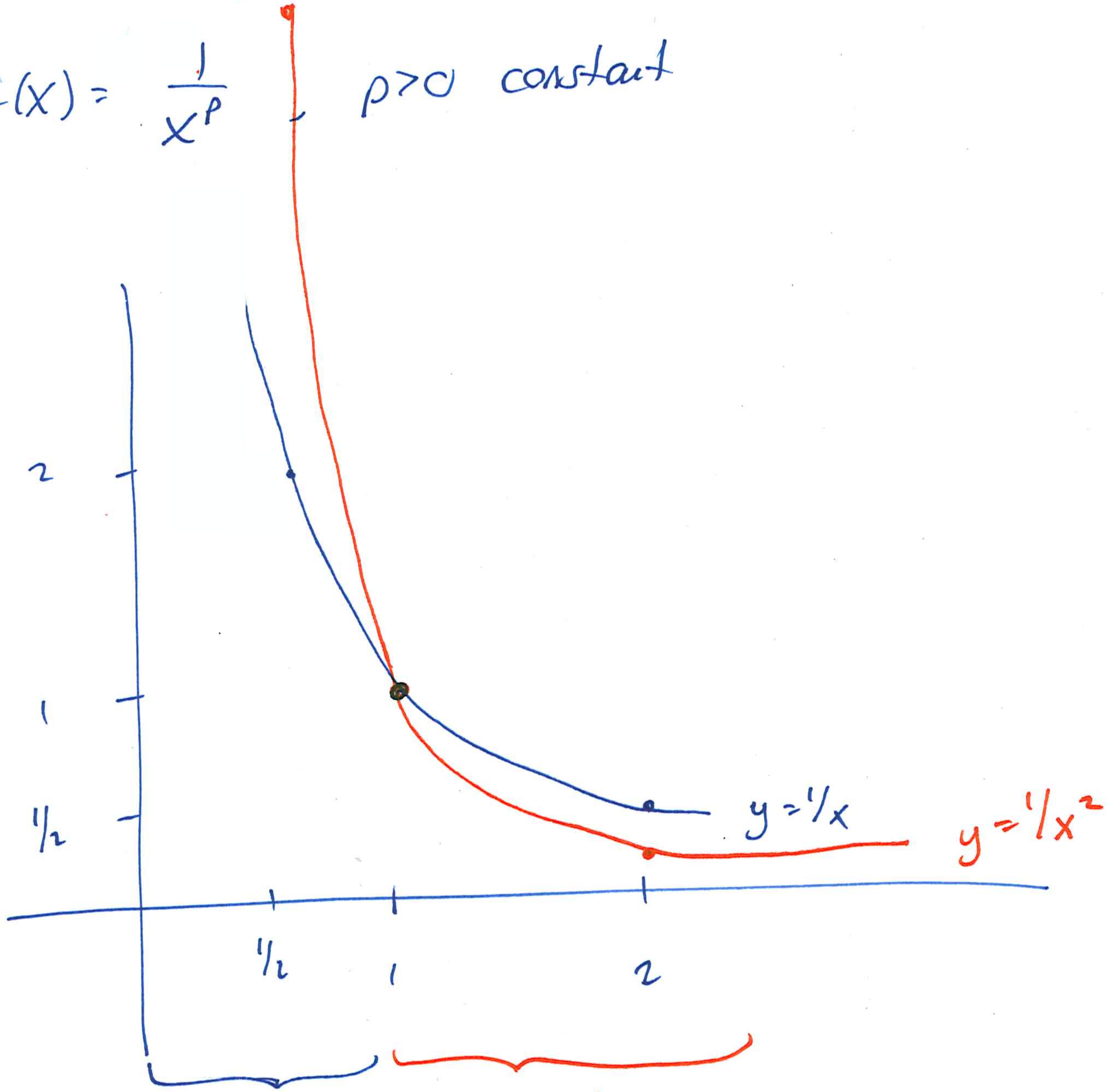
$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\left[\frac{-1}{x}\Big|_a^1\right] = \lim_{a\to 0^+}\left[\frac{-1}{1} - \underbrace{\frac{-1}{a}}_{\to\infty}\right] = \infty$$

DIVERGES

$f(x) = \dfrac{1}{x^p}$, $p > 0$ constant

$\dfrac{1}{(1/2)^2} = \dfrac{1}{1/4} = 4$

$\dfrac{1}{2^2} = \dfrac{1}{4}$



$\dfrac{1}{x} < \dfrac{1}{x^2}$ $\qquad$ $\dfrac{1}{x} > \dfrac{1}{x^2}$

ex $\int_0^1 \frac{1}{x^{0.999}}\,dx = \lim_{a\to 0^+}\left[\int_a^1 x^{-0.999}\,dx\right] =$

$\lim_{a\to 0^+}\left[1000\,x^{0.001}\Big|_a^1\right] = \lim_{a\to 0^+}\left[1000 - 1000\cdot \underbrace{\cancel{a^{0.001}}}_{0}\right] = 1000$

So: $\int_0^1 \frac{1}{x^{0.999}}\,dx$ converges

ex $\int_0^1 \frac{1}{x^{1.001}}\,dx = \lim_{a\to 0^+}\left[\int_a^1 x^{-1.001}\,dx\right] = \lim_{a\to 0^+}\left[-1000x^{-0.001}\Big|_a^1\right]$

$= \lim_{a\to 0^+}\left[-1000 + 1000\,a^{-0.001}\right]$

$= \lim_{a\to 0^+}\left[-1000 + \frac{1000}{a^{0.001}}\right]_{\to 0} = \infty$

$0.001 = \frac{1}{1000}$

$\frac{1}{0.001} = \frac{1}{1/1000} = 1000$

So: $\int_0^1 \frac{1}{x^{1.001}}\,dx$ DIVERGES p-test :

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{converges} & \text{if } p < 1 \\ \text{diverges} & \text{if } p \geq 1 \end{cases}$$

p positive constant

$$\int_1^\infty \frac{1}{x^p}\,dx : \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$$

ex $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$: DIVERGES (by p-test)

$p = 1/2 < 1$

ex $\int_0^1 \frac{1}{x^{2.7}}\,dx$ DIVERGES

$p = 2.7 > 1$

ex $\displaystyle\int_0^{17} \frac{1}{\sqrt{x}}\,dx$ Conv or Div?

$$= \underbrace{\int_0^{1} \frac{1}{\sqrt{x}}\,dx}_{\text{determines conv/div}} + \underbrace{\int_1^{17} \frac{1}{\sqrt{x}}\,dx}_{\text{some number (not improper)}}$$

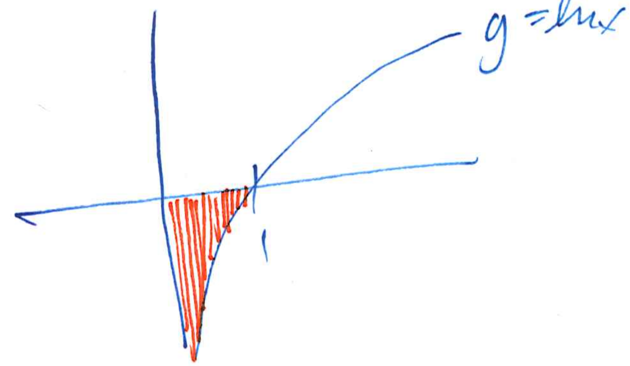
$p = 1/2 < 1$

converge

So: $\displaystyle\int_0^{17} \frac{1}{\sqrt{x}}\,dx$ converges.

LOL nevermind

ex $\int_0^1 \ln x \, dx$



Evaluate.

$$\int 1\cdot \ln x \, dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - \int 1\,dx$$

$$= \boxed{x\ln x - x} + C$$

$u: \ln x \qquad du: \frac{1}{x}dx$

$dv: 1\,dx \qquad v: x$

$$\int_0^1 \ln x\,dx = \lim_{a\to 0^+}\left[\int_a^1 \ln x\,dx\right] = \lim_{a\to 0^+}\left[\left(1\overset{0}{\cancel{\ln 1}}-1\right)-(a\ln a - a)\right]$$

$$= \lim_{a\to 0^+}\left[-1 - a\ln a + a\right] = \lim_{a\to 0^+}\left[-1 - \underset{0}{a}\left(\underset{-\infty}{\ln a - 1}\right)\right]$$

indeterminate form

$$= \lim_{a\to 0^+}\left[-1 - \underbrace{\frac{\ln a - 1}{1/a}}_{\frac{\infty}{\infty}}\right] = \lim_{a\to 0^+}\left[-1 + \frac{1/a}{+1/a^2}\right] = \lim_{a\to 0^+}\left[-1 + a\right] = \boxed{-1}$$

use L'Hospital

Ch 7.9 Differential Equations

(ex) $\boxed{y' = e^x}$ and $y(0) = 2$. What is y?

1st order differential equation

$y = \int e^x dx = e^x + C$, $\quad y = e^x + C$

$x=0$: $\quad 2 = e^0 + C$

$2 = 1 + C$

$C = 1$

So: $\boxed{y = e^x + 1}$

2nd order differential equation

(ex) $\boxed{y''(t) = 12t + 1}$, $y(0) = 1$, $y(1) = 10$. Find y

$y'(t) = \int 12t + 1 \, dt = 6t^2 + t + C$

$y(t) = \int (6t^2 + t + C) \, dt = 2t^3 + \frac{1}{2}t^2 + C\cancel{t} + D$

$y(t) = 2t^3 + \frac{1}{2}t^2 + Ct + D$

$1 = D \quad (y(0) = 1)$

$10 = 2 + \frac{1}{2} + C + D$

$10 = 2 + \frac{1}{2} + C + 1$

$C = 6.5$

$\boxed{y = 2t^3 + \frac{1}{2}t^2 + 6.5t + 1}$

(ex) $\boxed{y' = ky + b}$

where k, b constants

y function of ~~x~~ t

General Solution:

$$y = Ce^{kt} - b/k$$

for some C-constant

(ex) $y' = 3y + 7$, $y(2) = 5$

what is y?

$y = Ce^{3t} - 7/3$ for some C

[$y(2)=5$] $5 = C \cdot e^{3 \cdot 2} - 7/3$ find C

$\frac{22}{3} = C \cdot e^6$

$\frac{22}{3 \cdot e^6} = C$

So: $y = \frac{22}{3e^6} \cdot e^{3t} - 7/3$

$$\boxed{y = \frac{22}{3} e^{3t-6} - 7/3}$$

Check: $y' = 3y + 7$

$\underline{y'} = \frac{22}{3} \cdot e^{3t-6} \cdot 3 = \underline{\underline{22e^{3t-6}}}$

$\underline{3y+7} = 3(\underbrace{\tfrac{22}{3} e^{3t-6} - 7/3}_{y}) + 7$

$= 22e^{3t-6} - 7 + 7 = \underline{\underline{22e^{3t-6}}}$

TRUE: $y' = 3y + 7$ for this y

Check: $y(2) = 5$

$$y(t) = \frac{22}{3} e^{3t-6} - 7/3$$

$$y(2) = \frac{22}{3} e^{0} - 7/3 = \frac{22}{3} - \frac{7}{3} = \frac{15}{3} = 5 \checkmark$$

Differential Equations

Which of the following satisfies

↳ makes true

$$\frac{dy}{dx} + x^2 - 1 = y$$

(A) $y = x^2 + 1$ — $(2x) + x^2 - 1 = x^2 + 1$

$x^2 + 2x - 1 = x^2 + 1$ FALSE

not a solution

(B) $y = x^2 + 2x + 1$ — $(2x+2) + x^2 - 1 = x^2 + 2x + 1$

$x^2 + 2x + 1 = x^2 + 2x + 1$ TRUE

a solution to our diff. eq.

(C) $y = -\frac{1}{3}x^3 + x$ — $(-\cancel{x^2} + 1) + \cancel{x^2} - 1 = -\frac{1}{3}x^3 + x$

$0 = -\frac{1}{3}x^3 + x$ FALSE

not a solution

(ex) $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x+2$$

Which is a solution?

~~(A)~~ $y = e^x$

$$e^x + 2e^x + e^x = x+2$$
$$4e^x = x+2$$

FALSE

~~(B)~~ $y = x^2$
$y' = 2x$
$y'' = 2$

$$2 + 2(2x) + x^2 = x+2$$
$$x^2 + 4x + 2 = x+2$$

FALSE

✓ (C) $y = x$
$y' = 1$
$y'' = 0$

$$0 + 2(1) + x = x+2$$
$$2 + x = x+2$$

TRUE

~~(D)~~ $y = x+1$
$y' = 1$
$y'' = 0$

$$0 + 2\cdot(1) + (x+1) = x+2$$
$$x+3 = x+2$$

FALSE

Separable Differential Equations

ex $\frac{dy}{dx} = y^2 x, \quad y(0) = 1$

long

$$\frac{1}{y^2} \cdot \frac{dy}{dx} = x$$

$$\int \frac{1}{y^2} \frac{dy}{dx}\, dx = \int x\, dx$$

$$\frac{dy}{dx} = \frac{dy}{dx}$$

$$dy = \left(\frac{dy}{dx}\right) dx$$

$$\int \frac{1}{y^2}\, dy = \int x\, dx$$

$$\int \frac{1}{y^2} y'\, dx \rightarrow dy$$

$$y' = \frac{dy}{dx}$$

$$y'\, dx = dy$$

short

$$\frac{1}{y^2} \cdot \frac{dy}{dx} = x$$

$$\frac{1}{y^2}\, dy = x\, dx$$

$$\int \frac{1}{y^2}\, dy = \int x\, dx$$

$$\frac{-1}{y} = \frac{1}{2}x^2 + C$$

find C

If $x=0$, $y=1$

$$\frac{-1}{1} = 0 + C \rightarrow \boxed{C = -1}$$

find y

$$\frac{-1}{y} = \frac{1}{2}x^2 - 1$$

$$\frac{1}{y} = -\frac{1}{2}x^2 + 1$$

$$\boxed{y = \frac{1}{-\frac{1}{2}x^2 + 1}}$$

ex $\dfrac{dy}{dx} = e^{x-y}$

$dy = e^{x-y}\,dx$

$dy = \dfrac{e^x}{e^y}\,dx$

$e^y\,dy = e^x\,dx$

$\int e^y\,dy = \int e^x\,dx$

$e^y = e^x + C$

$\boxed{y = \ln(e^x + C)}$

ex $\dfrac{dy}{dx} = y(4x^3 - 1)$, $\boxed{y(0) = -2}$

$\frac{1}{y}\,dy = (4x^3 - 1)\,dx$

$\int \frac{1}{y}\,dy = \int (4x^3 - 1)\,dx$

$\ln|y| = x^4 - x + C$

Find C:

If $x = 0$, $y = -2$

$\ln|-2| = 0^4 - 0 + C$

$\ln 2 = C$

$\ln|y| = x^4 - x + \ln 2$

$|y| = e^{x^4 - x + \ln 2}$

$\boxed{y = -e^{x^4 - x + \ln 2}}$

Find y

If $y > 0$, $|y| = y$

If $y < 0$, $|y| = -y$

ex $\dfrac{dy}{dx} \cdot \sqrt{y(9-x^2)} = -2x$ where $y > 0$ for all x

Note:

$y = \sqrt[3]{3\sqrt{9-x^2}+c}^{\,2}$

$\dfrac{dy}{dx} \cdot \sqrt{y} \cdot \sqrt{9-x^2} = -2x$

$\int \sqrt{y}\, dy = \int \dfrac{-2x}{\sqrt{9-x^2}}\, dx$

$\int y^{1/2}\, dy = \int \dfrac{-2x}{\sqrt{9-x^2}}\, dx$ du

$u = 9-x^2$

$du = -2x\, dx$

$\frac{2}{3} y^{3/2} = \int \frac{1}{\sqrt{u}}\, du$

$\frac{2}{3} y^{3/2} = \int u^{-1/2}\, du$

$\frac{2}{3} y^{3/2} = 2u^{1/2} + C$

$\frac{2}{3} y^{3/2} = 2\sqrt{9-x^2} + C$

$y^{3/2} = 3\sqrt{9-x^2} + C$

$\boxed{y = [\,3\sqrt{9-x^2} + C\,]^{2/3}}$

ex $\sec x \dfrac{dy}{dx} = y^3$

$$\frac{1}{\cos x} \cdot \frac{dy}{dx} = y^3$$

$$\int y^{-3}\,dy = \int \cos x\,dx$$

$$-\frac{1}{2} y^{-2} = \sin x + C$$

$$\frac{-1}{2y^2} = \frac{\sin x + C}{1}$$

$$\frac{2y^2}{-1} = \frac{1}{\sin x + C}$$

$$2y^2 = \frac{-1}{\sin x + C}$$

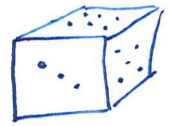
$$y^2 = \frac{-1}{2\sin x + C}$$

① $y = \sqrt{\dfrac{-1}{2\sin x + C}}$

$\boxed{\text{OR}}$

② $y = -\sqrt{\dfrac{-1}{2\sin x + C}}$

$\boxed{\text{Probability}}$



- Probability: Number from 0 to 1
  Interpret: liklihood an event will happen

  0: no matter how many tries, never happens

  1: (100%) no many tries, always happens

  1/3: if try lots & lots of times, event happens in ~1/3

  $\left(\lim_{\#\text{tries}\to\infty} \left[\frac{\#\text{ tries where event happened}}{\#\text{ tries total}} \right] \right)$

Notation: (conventional)

X: dice roll

Event (trial): capital letter, X

x: 4

Value event might take: lower-case letter, x

* $\boxed{Pr(X=x)}$: Probability that trial X gives a value of x

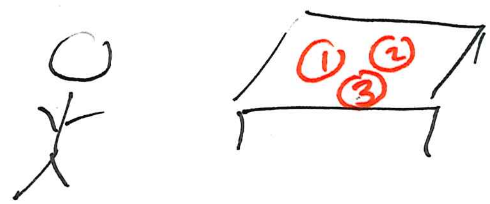
X: rolling a dice

$$\underbrace{Pr(X=6)}_{\text{Prob. that I roll a 6}}$$

$Pr(X=1)$

ex

X: selection of participant



1: your product

2: competitors'

3: products

What is $\underbrace{Pr(X=1)}_{\text{probability that selection of participant is 1}}$

$$Pr(X = x \quad \text{or} \quad X \neq x) = 1$$

$$Pr(X = x) = 1 - Pr(X \neq x)$$

(ex) If an unfair coin flips Heads 70% of the time

Then: $Pr(X = H) = 0.7$

$Pr(X = T) = 0.3$

discrete : "listable"
possible outcomes of an event

- Roll 3 dice, add values

Outcomes : 3, 4, 5, ..., 18

DISCRETE

- Choose a whole # from 1 to 10

Outcomes : 1, 2, 3, ..., 10

DISCRETE

- Choose any real # from 1 to 10

Outcomes : [1, 10]

exist along a continuum

NOT DISCRETE

CONTINUOUS

- The exact age of a person at noon today

Outcom: [0, 200]

NOT DISCRETE

- Amount of oil spilled in an oil spill

Ambiguous

weight could be any #, [0, N]
N : weight of earth
(not discrete)

molecules
whole # (discrete)

Def: A continuous random variable is one that has a continuous Cumulative Distribution Function (CDF)

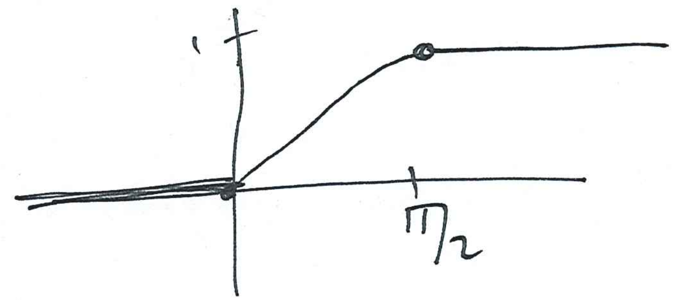
For any continuous random variable X,

$$Pr(X = x) = 0$$

So: $Pr(X \le x) = Pr(X < x)$

(ex) Suppose X is a continuous random variable and its CDF is:

$$F(x) = \begin{cases} 0 & x < 0 \\ \sin x & 0 \le x \le \pi/2 \\ 1 & x > \pi/2 \end{cases}$$



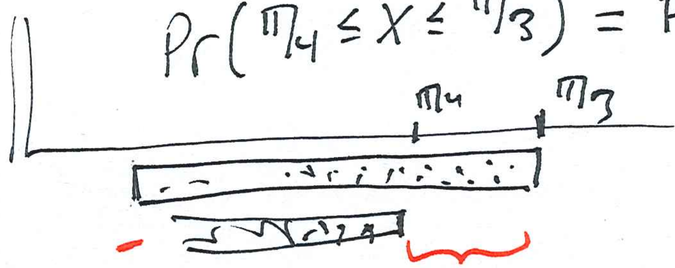
$Pr(X \le 0) = F(0) = 0$

$Pr(X \le 1) = \sin(1)$

$$Pr(\pi/4 \le X \le \pi/3) = F(\pi/3) - F(\pi/4)$$

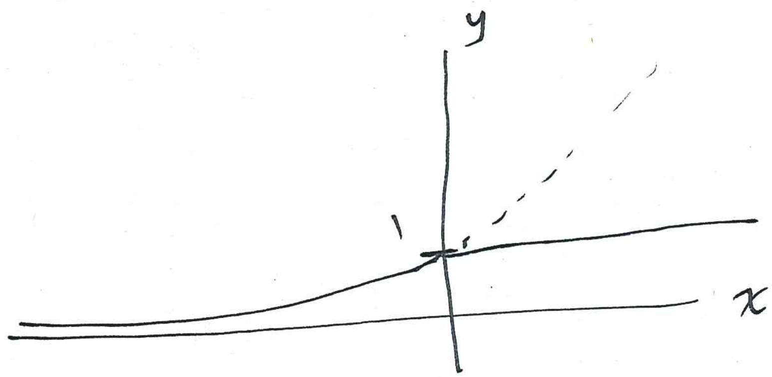
$= \sin(\pi/3) - \sin(\pi/4)$

$= \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} = \frac{\sqrt{3}-\sqrt{2}}{2}$



(ex) X is a random variable w/ cumulative distribution function

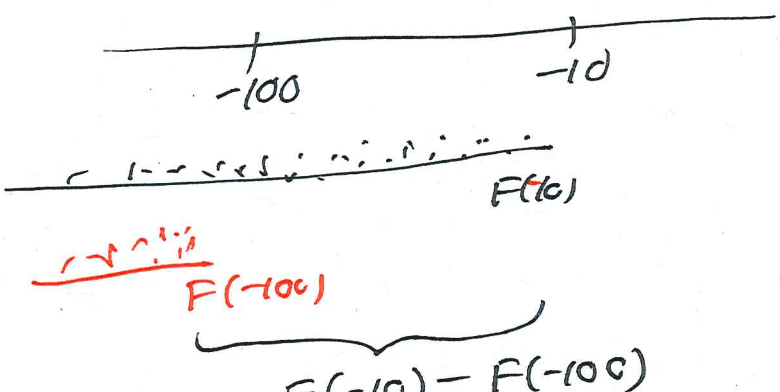
$$F(x) = \begin{cases} e^{x} & x \le 0 \\ 1 & x > 0 \end{cases}$$



$Pr(X \le 0) = F(0) = e^{0} = 1$

$Pr(X < 1) = Pr(X \le 1) = F(1) = 1$

$Pr(-100 \le X \le -10) = F(-10) - F(-100) = e^{-10} - e^{-100} = \frac{1}{e^{10}} - \frac{1}{e^{100}}$



If $F(x)$ is a CDF:

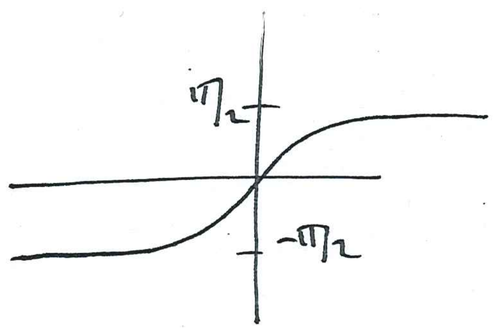
$F(b) - F(a) = Pr(a \le X \le b)$

$Pr(X \ge a) = Pr(X \not< a)$

$= 1 - F(a)$

(ex) Suppose $F(x) = k\arctan x + c$ for some constants k, c

If $F(x)$ is a Cumulative Distribution Function, what are k & c?



$y = \arctan x$

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty}\left[k\arctan x + c\right]$$

$$= k(\pi/2) + c = 1$$

So: $\boxed{c = 1 - k(\pi/2)}$

$$\lim_{x\to -\infty} F(x) = \lim_{x\to -\infty}\left[k\arctan x + c\right]$$

$$= k(-\pi/2) + c = 0$$

$\boxed{c = k(\pi/2)}$

$$1 - k(\pi/2) = k(\pi/2)$$

$$1 = 2k(\pi/2) = k\pi$$

$\boxed{k = 1/\pi}$

$$c = k(\pi/2) = \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2}$$

$\boxed{c = 1/2}$

Cumulative Distribution Function

T : temp outside (tomorrow)

Reasonable questions:

$F(0) = Pr(T \le 0)$

$F(20) = Pr(T \le 20)$

$F(50) = Pr(T \le 50)$

Want a function

$F(x) = Pr(T \le x)$

Properties of $F(x)$

$F(x)$ probability $\Rightarrow$ $0 \le F(x) \le 1$ for any x

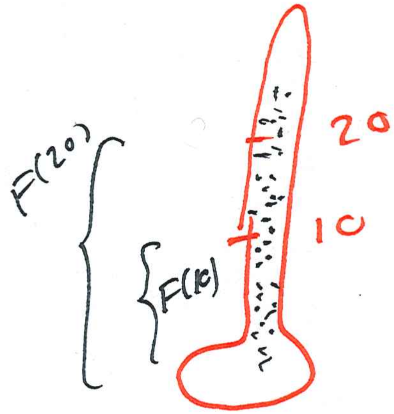
$F(1000) = Pr(T \le 1000) = 1$

$F(-1000) = Pr(T \le -1000) = 0$

$\lim_{x \to \infty} F(x) = 1$

$\lim_{x \to -\infty} F(x) = 0$

Compare $F(10)$, $F(20)$



$F(10) = Pr(T \le 10)$: proportion of days when $T \le 10$

$F(20) = Pr(T \le 20)$: days when $T \le 20$

So: $F(10) \le F(20)$

$F(x)$ is nondecreasing

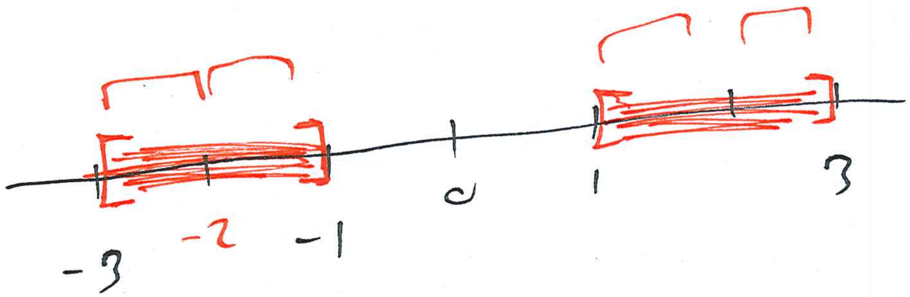
Def: The cumulative distribution function (CDF) of a random variable X is:

$$F(x) = Pr(X \le x)$$

- $0 \le F(x) \le 1$
- $\lim\limits_{x \to \infty} F(x) = 1$
- $\lim\limits_{x \to -\infty} F(x) = 0$
- $F(x)$ is a nondecreasing function of x

[eg $F(0) \le F(1) \le F(2) \le F(2.5) \ldots$]

ex



Choose any number from $[-3,-1] \cup [1,3]$.
"uniformly" (no preference), call this event X

$F(x)$: cumulative distribution function of X

So: $F(x) = Pr(X \le x)$

$F(0) = Pr(X \le 0) = \frac{1}{2}$

$F(-2) = Pr(X \le -2) = \frac{1}{4}$

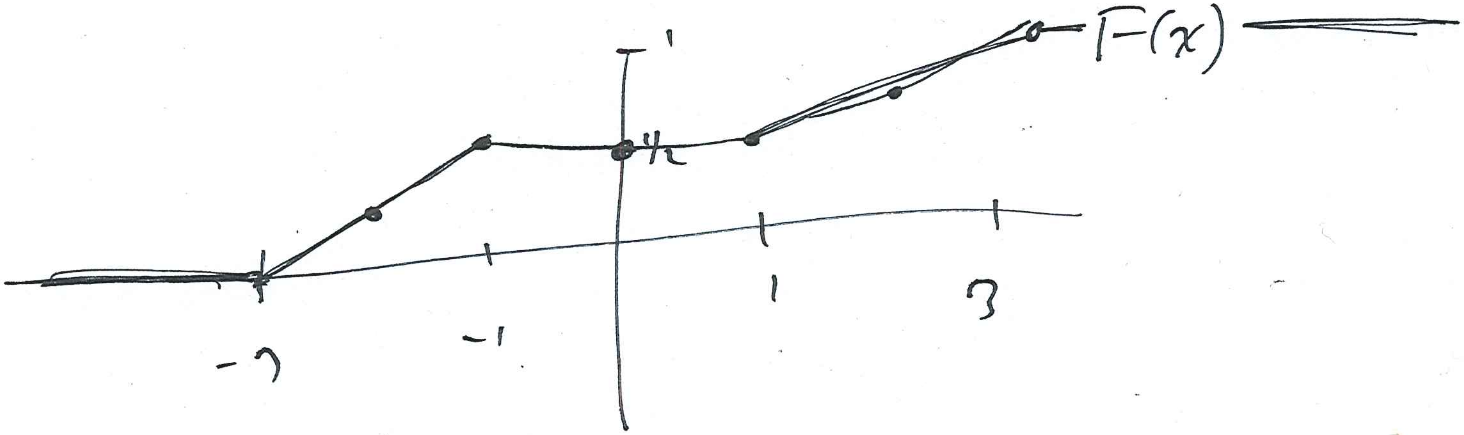
$F(2) = Pr(X \le 2) = 3/4$

$F(-1) = 1/2$

$F(1) = 1/2$

$Pr(X=1) = 0$

So: $Pr(X \le 1) = Pr(X < 1)$


F(x)
1
1/2
-2
-1
1
2

Probability Density Function

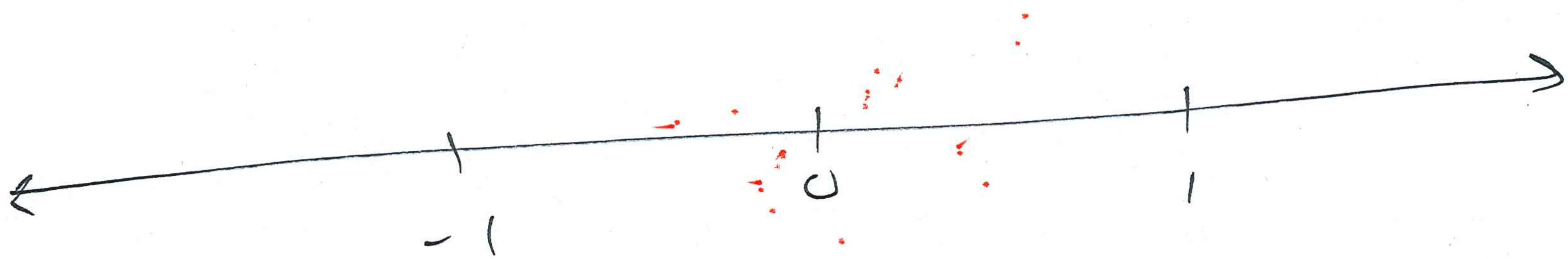
Problem: If X is a continuous random variable

then $Pr(X = x) = 0$

But not all "regions" may be equally likely

So - how do we describe "preference?"

=



Thought!

$$Pr(X \approx x) \approx \frac{Pr(x \leq X \leq x+h)}{h} = \frac{F(x+h) - F(x)}{h}$$ derivative!

small h

Def:

Let $F(x)$ be the cumulative distribution function of a cont. random variable X.

The probability density function (PDF) of x is:

$$f(x) = \frac{d}{dx}\{F(x)\}$$ (when it exists)

We often go the other way:

If $f(x)$ is the PDF to X,

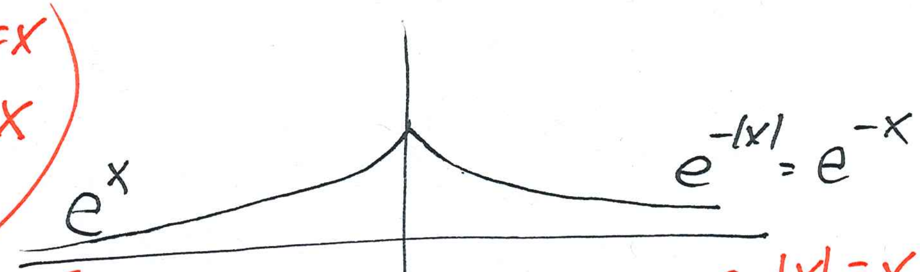
then $$\underbrace{F(x)}_{\text{CDF}} = \int_{-\infty}^{x} f(t)\,dt$$

(ex) Suppose $f(x) = ae^{-|x|}$ is a PDF (a constant)

- What is a?
- What is $Pr(-3 \le X \le 1)$?

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

If $x \ge 0$, $|x| = x$
If $x < 0$, $|x| = -x$



$x < 0$, so $|x| = -x$
$e^{-|x|} = e^{-(-x)} = e^x$

$x > 0$, so $|x| = x$
$e^{-|x|} = e^{-x}$

$$\int_{-\infty}^{\infty} a \cdot e^{-|x|}\,dx$$

$$= \int_{-\infty}^{0} a \cdot e^{-|x|}\,dx + \int_{0}^{\infty} a \cdot e^{-|x|}\,dx$$

$$\overset{\text{(even)}}{=} 2\int_{0}^{\infty} a \cdot e^{-|x|}\,dx \underset{x \ge 0}{=} 2\int_{0}^{\infty} a \cdot e^{-x}\,dx = 2a \lim_{b\to\infty} \int_{0}^{b} e^{-x}\,dx$$

$$= 2a \lim_{b\to\infty} \left[-e^{-x} \Big|_{0}^{b} \right] = 2a \left[\lim_{b\to\infty} \left(-e^{-b} - -e^{-0} \right) \right]$$

$$= 2a \lim_{b\to\infty} \left[\underbrace{\frac{-1}{e^{b}}}_{0} + 1 \right] = 2a = 1$$

So: $\boxed{a = 1/2}$

$f(x)$: Prob. Density Function

$$F(b) - F(a) = \int_{-\infty}^{b} f(t)dt - \int_{-\infty}^{a} f(t)dt = \underbrace{\int_a^b f(t)dt}_{\text{useful!}}$$

$$\| $$

$$Pr(a \le X \le b)$$

So: $\int_a^b f(t)dt$ is some probability whenever $a < b$

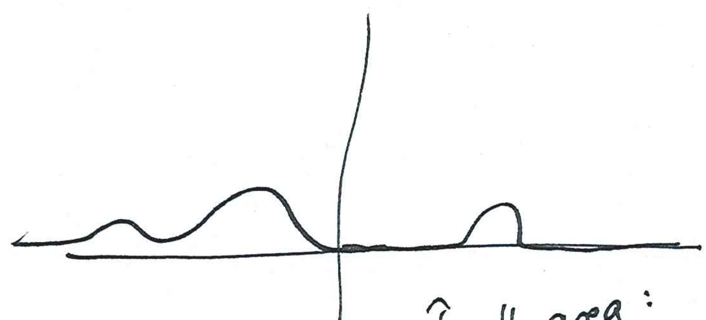
so: $0 \le \int_a^b f(t)dt$

So: $\boxed{f(x) \geq 0 \text{ for all } x}$

Also: $\boxed{\int_{-\infty}^{\infty} f(x)dx = 1}$

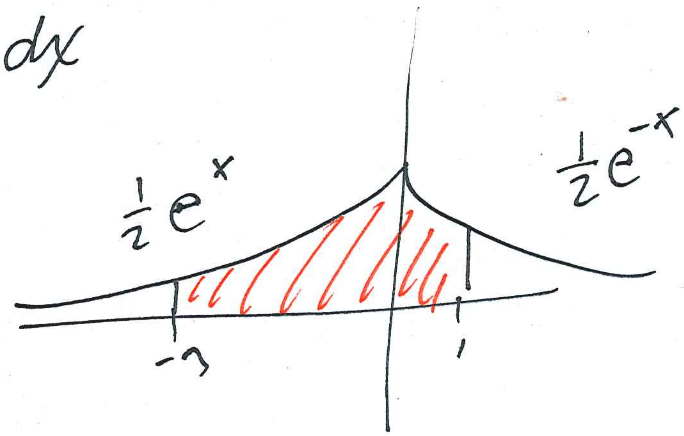
$$\|$$

$$Pr(-\infty \le X \le \infty)$$



all area: 1

$$\Pr(-3 \le x \le 1) = \int_{-3}^{1} f(x)\,dx$$



$$= \int_{-3}^{1} \frac{1}{2} e^{-|x|}\,dx$$

$$= \int_{-3}^{0} \frac{1}{2} e^{-|x|}\,dx + \int_{0}^{1} \frac{1}{2} e^{-|x|}\,dx$$

$$= \int_{-3}^{0} \frac{1}{2} e^{x}\,dx + \int_{0}^{1} \frac{1}{2} e^{-x}\,dx$$

$$= \left(\frac{1}{2}e^{x}\Big|_{-3}^{0}\right) + \left(\frac{-1}{2}e^{-x}\Big|_{0}^{1}\right)$$

$$= \left(\frac{1}{2}e^{0} - \frac{1}{2}e^{-3}\right) + \left(\frac{-1}{2}e^{-1} - \frac{-1}{2}e^{0}\right)$$

$$= \left(\frac{1}{2} - \frac{1}{2e^{3}}\right) + \left(\frac{-1}{2e} + \frac{1}{2}\right)$$

$$= \boxed{1 - \frac{1}{2e^{3}} - \frac{1}{2e}}$$

ex $$f(x) = \begin{cases} k(3x^2+1) & \text{if } 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

- Find the value of k that makes f a Probability Density Function.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$ ← Solve for k

- $Pr(1 \le X \le 2) = \int_1^2 f(x)\,dx$ ← calculate

- Find the cumulative distribution function of X

$$F(x) = Pr(X \le x) = Pr(-\infty \le X \le x) = \int_{-\infty}^{x} f(t)\,dt$$ ↑ calculate

$$= \int_{-\infty}^{x} k(3t^2+1)\,dt$$

- Find the probability that $X = 1$. (0)

$$\int_1^1 f(x)\,dx = 0$$

Expected Value of a Continuous Random Variable

Discrete Case:

Homework: 70, 70, 70, 80, 80

Average: $\dfrac{70+70+70+80+80}{5} = \dfrac{3(70)+2(80)}{5} = \dfrac{3}{5}(70)+\dfrac{2}{5}(80)$

prob. that that value happened

value

value

Continuous Random Variable

$$\int x \cdot f(x)\,dx$$

add up

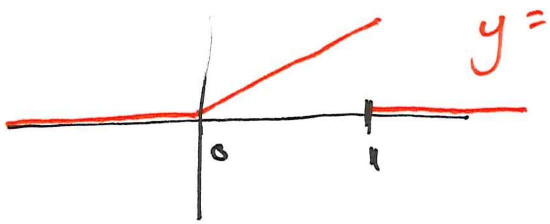
value

probability density

Def: The expected value ("expectation," "mean") of a continuous random variable X, with probability density function $f(x)$, is

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

(ex) Suppose X has PDF $f(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$



$y = f(x)$

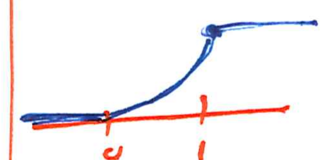
$$\int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_{-\infty}^{0} x \cdot \underbrace{0}_{} \, dx + \underbrace{\int_0^1 x \cdot \underbrace{2x}_{} \, dx}_{} + \int_1^{\infty} x \cdot \underbrace{0}_{} \, dx$$

"0" "0"

$$= \int_0^1 2x^2\, dx = \frac{2}{3}x^3 \Big|_0^1 = \frac{2}{3} - 0 = \boxed{\frac{2}{3}} = \mathbb{E}(X)$$

$F(x)$ continuous

$F(x) = \int_{-\infty}^{x} f(t)\,dt$



CDF

(ex) X is a continuous random variable with Cumulative Distribution Function is

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2 + \frac{3}{2}x & 0 \le x \le \frac{1}{2} \\ 1 & x > 1/2 \end{cases}$$

$F(1/2) = 1$:
$Pr(X \le 1/2) = 100\%$.
X always $\le 1/2$

What is $\mathbb{E}(X)$?

Recall:
$f(x) = F'(x)$
PDF

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$f(x) = \begin{cases} 0 & x < 0 \\ 2x + 3/2 & 0 < x < 1/2 \\ 0 & x > 1/2 \end{cases}$$

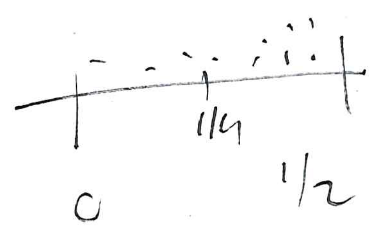
$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$= 0 + \int_0^{1/2} x(2x + 3/2)\,dx + 0$$

$$= \int_0^{1/2} 2x^2 + \tfrac{3}{2}x\,dx = \tfrac{2}{3}x^3 + \tfrac{3}{4}x^2 \Big|_0^{1/2}$$

$$= \frac{2}{3} \cdot \frac{1}{2^3} + \frac{3}{4} \cdot \frac{1}{2^2} = \frac{1}{3 \cdot 2^2} + \frac{3}{2^4} = \frac{4}{3 \cdot 2^4} + \frac{9}{3 \cdot 2^4}$$

$$= \frac{13}{3 \cdot 16} = \frac{13}{48} > \frac{1}{4}$$



Variance & Standard Deviation

Motivation:

HW1 : 50 50 50 50 Avg: 50

HW2 : 0 0 100 100 Avg: $\frac{1}{2}(0)+\frac{1}{2}(100)=50$

On average, how close is everyone to average?

Idea #1: Calculate avg of $(x-\underbrace{50}_{avg})$

HW1:

grade	g-avg
50	0
50	0
50	0
50	0

Avg: 0

HW2:

$\boxed{\text{Idea \#2}}$

grade	g-avg	$(g-avg)^2$
0	-50	50^2
0	-50	50^2
100	+50	50^2
100	+50	50^2

Avg: 0 Avg: 50^2

Fix it! $\sqrt{50^2}=50$

X : continuous random variable

$\mathbb{E}(X)$: mean, expectation

$x - \mathbb{E}(X)$: how far value x is from $\mathbb{E}(X)$

$(x - \mathbb{E}(X))^2$: destroy +/–

↑ take Expectation

$$\int_{-\infty}^{\infty} (x - \mathbb{E}(X))^2 \cdot f(x)\, dx$$ — VARIANCE

$$\sqrt{\int_{-\infty}^{\infty} (x - \mathbb{E}(X))^2 \cdot f(x)\, dx}$$ — STANDARD DEVIATION

Def: The variance of a continuous random variable X, with probability density function $f(x)$, is:

$$Var(X) = \int_{-\infty}^{\infty} (x - \mathbb{E}(X))^2 \cdot f(x)\, dx$$

$$= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

rearranging

The standard deviation of X is

$$\sigma(X) = \sqrt{Var(X)}$$

sigma

ex PDF from before:

$$f(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

We already calculated:

$$\mathbb{E}(X) = 2/3$$

Find: $Var(X)$, $\sigma(X)$

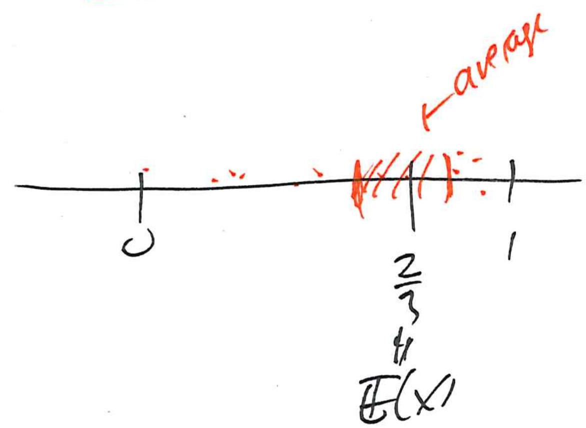
$$Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2 \cdot 2x\,dx = \int_0^1 2x^3\,dx = \frac{1}{2}x^4 \Big|_0^1$$

$$= \frac{1}{2}$$

$$Var(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \boxed{\frac{1}{18}}$$

$$\sigma(X) = \sqrt{\frac{1}{18}} = \boxed{\frac{1}{3\sqrt{2}}}$$



Q: St dev is $\sqrt{\text{Variance}}$

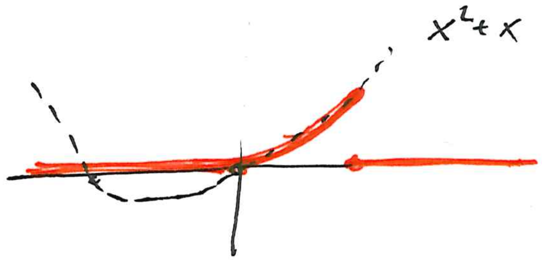
What do if $Var(X) < 0$? ← never happens!

$$Var(X) = \int_{-\infty}^{\infty} \underbrace{(x - E(X))^2}_{\geq 0} \cdot \underbrace{f(x)}_{\geq 0} \, dx$$

(ex) The length of time X used by students to complete a 1-hour exam is a random variable, with PDF:

p. A-821 in appendix

$$f(x) = \begin{cases} k(x^2+x) & \text{if } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$



Recall: $Pr(a \le X \le b) = \int_a^b f(x)\,dx$

So: $1 = \int_{-\infty}^{\infty} f(x)\,dx$

(a) What is k?

$$1 = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_0^1 k(x^2+x)\,dx = k\left(\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2\right)\Big|_0^1 = k\left(\tfrac{1}{3}+\tfrac{1}{2}\right) = k\left(\tfrac{5}{6}\right)$$

$$1 = k(5/6) \qquad \boxed{k = 6/5}$$

(b) Find the Cumulative Distribution Function

Recall: $F(x) = Pr(X \leq x)$

$= Pr(-\infty \leq X \leq x)$

$= \int_{-\infty}^{x} f(t)dt$

$\int_{-\infty}$

$$F(x) = \int_{-\infty}^{x} f(t)dt = \begin{cases} 0 & \text{if } x < 0 \\ (\frac{1}{3}x^3 + \frac{1}{2}x^2)6/5 & 0 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

(d) Find the probability that a randomly selected student will finish the exam in less than half an hour.

$$Pr(X \leq \tfrac{1}{2}) = F(1/2) = \tfrac{6}{5}\left(\tfrac{1}{3}\cdot\tfrac{1}{2^3} + \tfrac{1}{2}\cdot\tfrac{1}{2^2}\right)$$

(e) Find the expected time to complete the exam.

$\mathbb{E}(X) =$

$$\int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_0^1 x \cdot \frac{6}{5}(x^2+x)\,dx$$

$$= \int_0^1 \frac{6}{5}(x^3+x^2)\,dx = \frac{6}{5}\left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right)\Big|_0^1$$

$$= \frac{6}{5}\left(\frac{1}{4}+\frac{1}{3}\right) = \frac{6}{5}\left(\frac{7}{12}\right) = \frac{7}{10}$$ (42 min)

(f) Find Var(X), $\sigma(X)$

$$Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^1 x^2 \cdot \frac{6}{5}(x^2+x)\,dx$$

$$= \int_0^1 \frac{6}{5}(x^4+x^3)\,dx = \frac{6}{5}\left(\frac{1}{5}x^5 + \frac{1}{4}x^4\right)\Big|_0^1 = \frac{6}{5}\left(\frac{1}{5}+\frac{1}{4}\right)$$

$$= \frac{6}{5}\left(\frac{5+4}{20}\right) = \frac{6 \cdot 9}{100} = \frac{54}{100}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

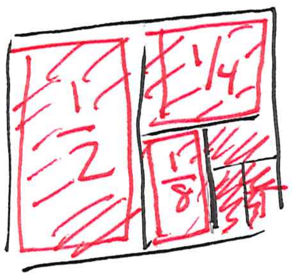
$$= \frac{54}{100} - \left(\frac{7}{10}\right)^2 = \frac{54-49}{100} = \frac{5}{100}$$

$$\sigma(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{\frac{5}{100}} = \boxed{\frac{\sqrt{5}}{10}}$$

↑ standard deviation

Sequences + Series

(ex)



Area of figure: 1

Areas of pieces:

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots$

list of #s (order)

"sequence"

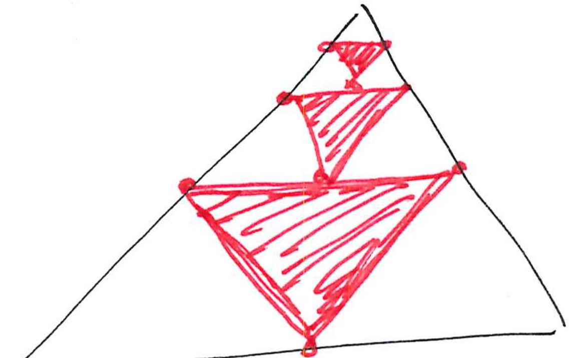
Sum areas of pieces:

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots = 1$

"series"

- sum of terms in a sequence

ex



equilateral triangle

Area: 1

Area of Pieces:

$\frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \ldots$

sequence

Sum of pieces:

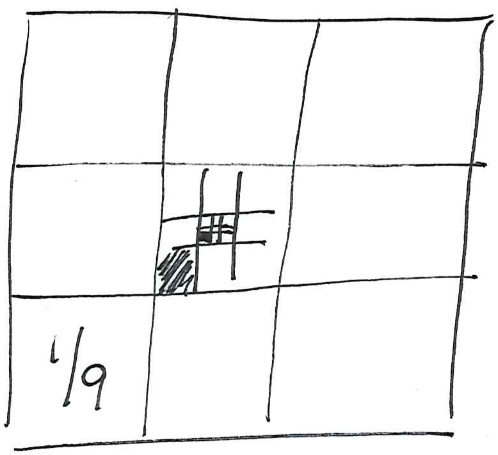
$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots = \frac{1}{3}$

series (sum)

ex

Sequence: $\frac{1}{9}, \frac{1}{9^2}, \frac{1}{9^3}, \frac{1}{9^4}, \dots$

Series: $\frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \frac{1}{9^4} + \dots = \frac{1}{8}$



We don't always want to use a sequence as a series

<u>Sequence</u> Ordered list of numbers

A function whose domain is whole numbers

$f(1) = 3,689,257$

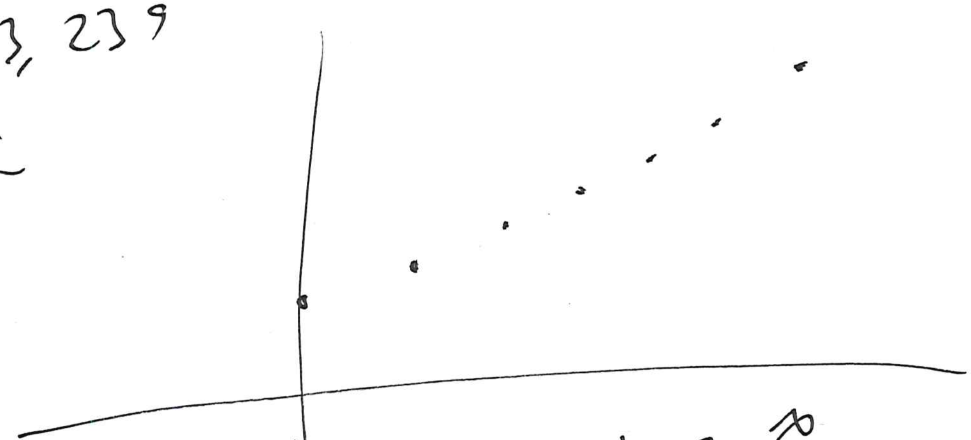
$f(2) = 4,324,810$

$f(3) = 4,833,239$

etc

$f(1/2)$ – no data

$f(-1)$ – no data



We write: $\{a_n\}_{n=1}^{\infty} = \{\frac{1}{2^n}\}_{n=1}^{\infty}$

each term

rule

$a_1 = \frac{1}{2}$, $a_2 = \frac{1}{2^2}$, $a_3 = \frac{1}{2^3}$, etc

Recursive description

depends on previous terms

Explicit description

you can find a term without knowing what came before

ex $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \cdots$

$a_n = \frac{1}{2^n}$ } explicit

$a_1 = \frac{1}{2}$

If $n>1$, $a_n = \frac{1}{2} a_{n-1}$ } recursive

term I want

term before

eg: $a_4 = \frac{1}{16}$

So: $a_5 = \frac{1}{2}\left(\frac{1}{16}\right) = \frac{1}{32}$

So: $a_6 = \frac{1}{2}\left(\frac{1}{32}\right) = \frac{1}{64}$

ex Fibonacci :

1, 1, 2, 3, 5, 8, 13, 21, ...

$$\begin{matrix} 1, & 1, & 2, & 3, & 5 \\ \| & \| & \| & \| & \| \\ F_0 & F_1 & F_2 & F_3 & F_4 \end{matrix}$$

Rule (recursive)

$F_0 = 1$

$F_1 = 1$

if $n > 1$, $F_n = F_{n-1} + F_{n-2}$

$F_2 = F_1 + F_0 = 1 + 1 = 2$

$F_3 = F_2 + F_1 = 2 + 1 = 3$

ex $\frac{-1}{3}, \frac{1}{9}, \frac{-1}{27}, \frac{1}{81}, \ldots$

(explicit)

- $\{a_n\}_{n=1}^{\infty} = \{(\frac{-1}{3})^n\}_{n=1}^{\infty}$

(recursive)

$a_1 = \frac{-1}{3}$

- For $n>1$, $a_n = \frac{-1}{3} \cdot a_{n-1}$

$= \frac{1}{2+1}, \frac{1}{4+1}, \frac{1}{8+1}, \frac{1}{16+1}, \frac{1}{32+1}, \ldots$

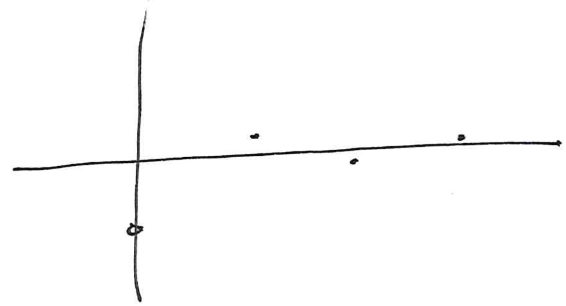
ex $\frac{1}{3}, \frac{1}{5}, \frac{1}{9}, \frac{1}{17}, \frac{1}{33}, \ldots$

(explicit)

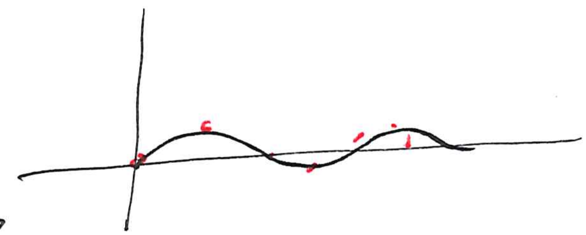
$\{a_n\}_{n=1}^{\infty} = \{\frac{1}{2^n+1}\}_{n=1}^{\infty}$

We can take limit as $n \to \infty$ of sequence

ex) $a_n = \left(\frac{-1}{3}\right)^n$



$\lim_{n \to \infty} a_n = 0$



$\boxed{\text{Theorem}}$

Let $f(x)$ be a function,

$\{a_n\}_{n=1}^{\infty} = \{f(n)\}_{n=1}^{\infty}$

If $\lim_{n \to \infty} f(n) = L$ for some real number L, then

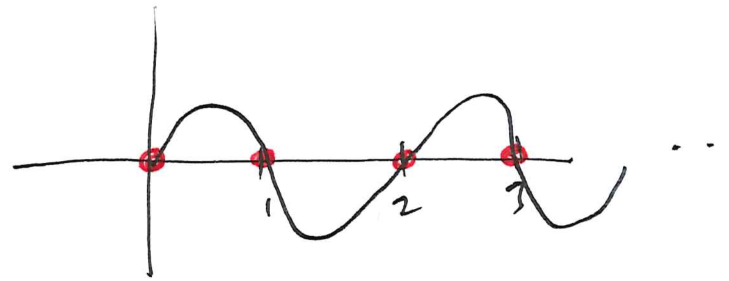
$\lim_{n \to \infty} a_n = L$

ex $\quad a_n = \dfrac{n^2+1}{n^2+2}$ $\qquad \dfrac{1+1}{1+2}, \dfrac{4+1}{4+2}, \dfrac{9+1}{9+2}, \ldots$

$$\lim_{n\to\infty} a_n = 1$$

Let $f(x) = \dfrac{x^2+1}{x^2+2}$. Then $\displaystyle\lim_{x\to\infty} \frac{x^2+1}{x^2+2} = 1$

ex $\quad$ Let $f(x) = \sin(\pi x)$



$$\lim_{x\to\infty} f(x) = \text{DNE}$$

$a_n = f(n)$

So: $\displaystyle\lim_{n\to\infty} a_n = 0$

Sequence: $\quad f(1), f(2), f(3), f(4), \ldots$

$\sin(\pi), \sin(2\pi), \sin(3\pi), \sin(4\pi), \ldots$

$0, 0, 0, 0$

function $f(x)$

sequence $a_n = f(n)$

$$\lim_{x\to\infty} f(x)$$

convergent
a real # L

DNE

$$\lim_{n\to\infty} a_n = L$$

(Theorem)

$\lim_{n\to\infty} a_n$ may or may not exist

(might cherry-pick values)

Limit Laws (sequences)

eg $\lim_{n\to\infty}\left[\underbrace{\frac{n^2+1}{n^2+2}}_{\to 1} + \underbrace{\sin(\pi n)}_{\to 0}\right] = 1+1 = 1$

Assume a_n, b_n are sequences, $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$,

A, B real #s

1. $\lim_{n\to\infty}(a_n \pm b_n) = A \pm B$

2. $\lim_{n\to\infty} c a_n = cA$, c constant

3. $\lim_{n\to\infty} a_n b_n = AB$

4. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = A/B$ if $B \neq 0$

(ex) $\{a_n\} = \left\{ \frac{n^2+3}{2n^2+1} \right\}$ $\qquad \lim_{n\to\infty} a_n = \frac{1}{2}$

$\{b_n\} = \{2\arctan n\}$ $\qquad \lim_{n\to\infty} b_n = \pi$

So: $c_n = \dfrac{2\arctan n\,(n^2+3)}{2n^2+1}$

(Limit Law #3) $\qquad \lim_{n\to\infty} c_n = \frac{1}{2}\cdot\pi = \pi/2$

(ex) $a_n = \frac{1}{n^2}$: $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$ $\qquad \lim_{n\to\infty} a_n = 0$

$b_n = 2n^2$: $2, 8, 18, 32, \ldots$ $\qquad \lim_{n\to\infty} b_n = \infty$

Can't simply say $\lim_{n\to\infty} (a_n b_n) = 0\cdot\infty \leftarrow ???$

$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} \left(\frac{1}{\cancel{n^2}} \cdot 2\cancel{n^2}\right) = \lim_{n\to\infty} (2) = 2$

ex $\lim\limits_{n\to\infty} \left[\underbrace{\cancel{\sin(\pi n)}}_{0} + \underbrace{\cos(\pi n)}_{-1,\,1,\,-1,\,1,\,-1,\,1}\right]$ DNE

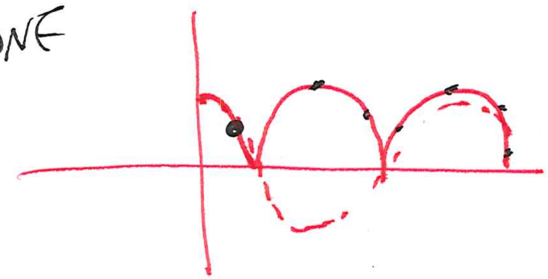
divergent sequence

Sequence: $-1, 1, -1, 1, -1, 1, -1, 1 \dots$

ex $\lim\limits_{n\to\infty} \left[\cos^2(n) + \sin^2(n)\right] = \lim\limits_{n\to\infty} [1] = 1$

$1, 1, 1, 1, 1, \dots$

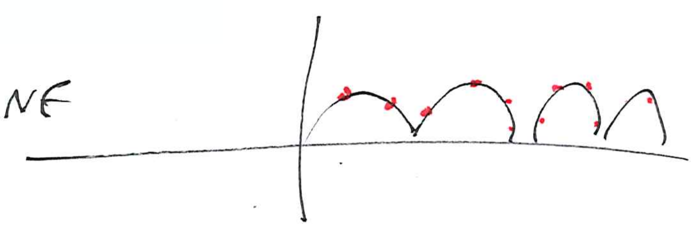
$\lim\limits_{n\to\infty} \cos^2 n$: DNE



$\cos^2 x$

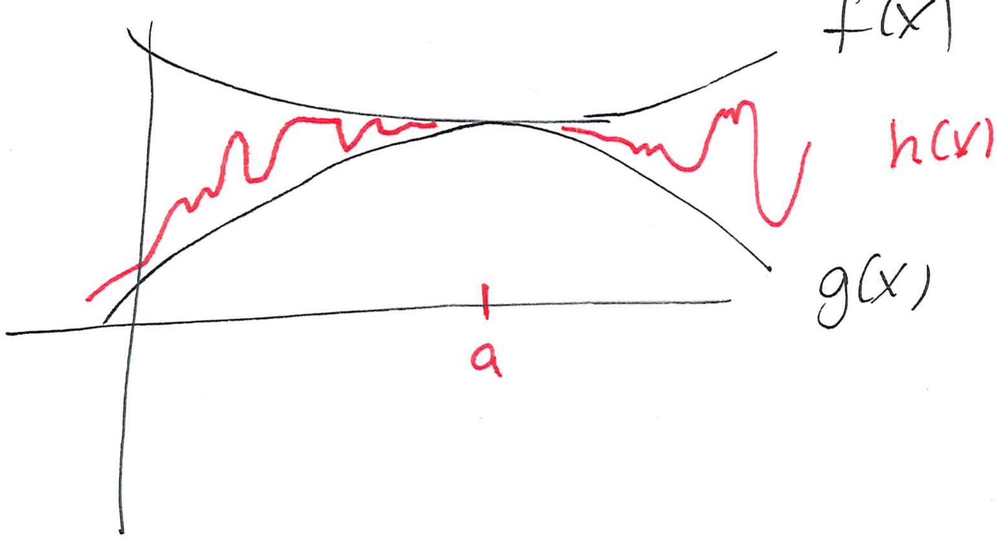
DIV

$\lim\limits_{n\to\infty} \sin^2 n$ DNE



$\sin^2 x$

Squeeze Theorem for Sequences



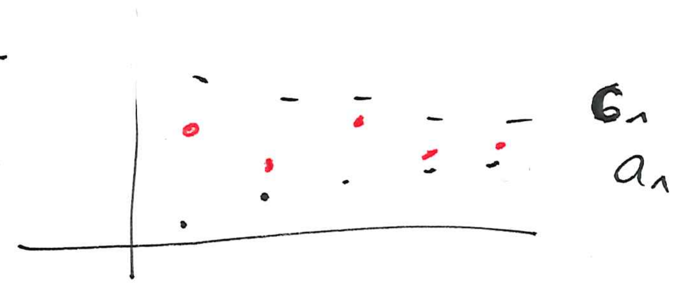
$g(x) \le h(x) \le f(x)$
for all x

If a_n, b_n, c_n are sequences, and:

- $a_n \le b_n \le c_n$ for all n larger than some value N

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

Then also $\lim_{n\to\infty} b_n = L$



ex $\{a_n\} = \left\{ \dfrac{2n + \cos n}{n+1} \right\}$

$$\underbrace{\frac{2n-1}{n+1}}_{b_n} \le \frac{2n+\cos n}{n+1} \le \underbrace{\frac{2n+1}{n+1}}_{c_n}$$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{2n+1}{n+1} = 2$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2n-1}{n+1} = 2$$

By Squeeze Theorem, $\lim_{n\to\infty} a_n = 2$.

Theorem

Every monotone, bounded sequence converges.

If a sequence a_n has $\lim_{n\to\infty} a_n = L$, where L is a real #

eg $\{a_n\} = \frac{1}{2^n}$ $\lim_{n\to\infty} a_n = 0$ ← real #

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

so $\{\frac{1}{2^n}\}$ converges

We say a sequence $\{a_n\}$ is bounded if there exist real #s L, U such that $L \le a_n \le U$ for all a_n.

eg $\{a_n\} = \frac{1}{2^n}$

$0 \le a_n \le 1$

A sequence is <u>monotone</u> if

- it never decreases

or

- it never increases

e.g. Sequence : $1, 2, 3, 4, 5, 6, 7, \ldots$ $\{a_n\} = \{n\}$

monotone (never goes down)

Sequence : $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ $\{a_n\} = \{\frac{1}{n}\}$

monotone (never goes up)

Sequence : $1, -1, 1, -1, 1, -1, \ldots$ $\{a_n\} = \{(-1)^{n+1}\}$

$\searrow \nearrow$

<u>not</u> monotone

example: Jar of 100 pieces of candy.
You never re-fill.
After n days, a_n be # of pieces in jar.

$\{a_n\}$: sequence
$a_0 = 100$

- Bounded : $0 \le a_n \le 100$
- Monotone : $\{a_n\}$ never increases

By theorem: $\{a_n\}$ converges (we don't know what $\lim_{n\to\infty} a_n$ is, but it is a real number)

Possibility: candy is gross
100, 99, 99, 99, 99, 99, 99, ... $\lim_{n\to\infty} a_n = 99$

Another possibility: candy is OK
100, 80, 80, 80, 20, 20, 20, 0, 0, 0, 0, 0 ... $\lim_{n\to\infty} a_n = 0$

Geometric Sequences

Ratio between consecutive terms is constant.

Ratio → $\div$

consecutive → next to each other

constant → always same

e.g. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

$\frac{1/4}{1/2} = \frac{1}{2}$ $\quad$ $\frac{1/8}{1/4} = \frac{1}{2}$ $\quad$ $\frac{1/16}{1/8} = \frac{1}{2}$ $\quad$ $\frac{1/32}{1/16} = \frac{1}{2}$

geometric sequence

eg Suppose in 2017, avg size of a hard drive is 500 GB, increases 40% every year.

Recall:

$X + (40\% \text{ of } x)$

$= X + 0.4X = 1.4X$

$a_0 = 500$

$a_1 = (1.4)500$

$a_2 = (1.4)(1.4)500$

$a_3 = (1.4)(1.4)(1.4)500$

Recursive:

$a_0 = 500$

$a_n = (1.4)a_{n-1}$

Explicit:

$a_n = 500(1.4)^n$

Geometric Sequences have the form:

$$a_n = r^n a_0$$
$$= r(a_{n-1})$$

ex $\{a_n\} = \{(\frac{2}{3})^n\}$ $\qquad \lim_{n\to\infty} a_n = 0$

Sequence: $\frac{2}{3}, (\frac{2}{3})^2, (\frac{2}{3})^3, \dots$

ex $\{a_n\} = \{1.1^n\}$ $\qquad \lim_{n\to\infty} a_n = \infty$

$1.1, (1.1)^2, (1.1)^3, (1.1)^4, \dots$

In general: $a_n = r^n$: $\quad \lim_{n\to\infty} a_n = \begin{cases} 0 & \text{if } |r|<1 \\ 1 & \text{if } r=1 \\ \text{DIV} & \text{if } |r|>1 \end{cases}$

$\{a_n\} = \{1^n\}$

$1, 1, 1, 1, 1, \dots$

Sequence of Partial Sums

Given a sequence a_n,

the sequence of partial sums S_n is:

$$S_n = a_1 + a_2 + \cdots + a_n$$

ex

some sequence a_n	add first n terms	sequence of partial sums. S_n
$a_1 =$ $1/2$	$1/2$	$1/2 = S_1$
$a_2 =$ $1/4$	$1/2 + 1/4 = 3/4$	$3/4 = S_2$
$a_3 =$ $1/8$	$1/2 + 1/4 + 1/8 = 7/8$	$7/8 = S_3$
$1/16$	$1/2 + 1/4 + 1/8 + 1/16 = 15/16$	$15/16 = S_4$
$\vdots$		
$a_n =$ $1/2^n$	$1/2 + 1/4 + 1/8 + \cdots + 1/2^n = 1 - \frac{1}{2^n}$	$1 - \frac{1}{2^n} = S_n$

$$\lim_{n\to\infty} a_n = 0$$

$$\lim_{n\to\infty} S_n = 1$$

What is $S_4 - S_3$?

$$\frac{15}{16} - \frac{7}{8} = \frac{15}{16} - \frac{14}{16} = \frac{1}{16} = a_4$$

$$S_4 - S_3 = [\cancel{a_1} + \cancel{a_2} + \cancel{a_3} + a_4] - [\cancel{a_1} + \cancel{a_2} + \cancel{a_3}] = a_4$$

What is $S_{15} - S_{14}$?

$$S_{15} - S_{14} = a_{15}$$

---

ex $\quad a_n = \frac{2}{10^n}, \qquad S_N = \sum_{n=1}^{N} a_n$

$a_1 = \frac{2}{10} = 0.2$

$a_2 = \frac{2}{10^2} = 0.02$

$a_3 = \frac{2}{10^3} = 0.002$

$\vdots$

$S_1 = 0.2$

$S_2 = 0.22$

$S_3 = 0.222$

$\vdots$

$$\lim_{n \to \infty} S_n = 0.\overline{22} = 2/9$$

$$\lim_{n \to \infty} a_n = 0$$

(ex) Suppose $\{S_N\}$ is the partial sum of $\{a_n\}$

$$S_N = \frac{1}{2^N}$$

Q: are values a_n positive or negative?

S_N decreasing, so we must be adding negative #s

$$S_N - S_{N-1} = a_N$$

eg $S_4 - S_3 = a_4$: $a_4 = \frac{1}{2^4} - \frac{1}{2^3}$

$(a_1 + a_2 + a_3 + a_4) - (a_1 + a_2 + a_3) = a_4$

$S_1 = a_1$

So: $a_1 = S_1 = \frac{1}{2}$

After that, all a_n negative.

Geometric Partial Sums

$\{a_n\}_{n=0}^{\infty}$: $\quad 1, r, r^2, r^3, r^4, \ldots$

$$S_N = \sum_{n=0}^{N} a_n = (1 + r + r^2 + \cdots + r^N)$$

$$S_{N+1} = (\underbrace{1 + r + r^2 + \cdots + r^N}) + r^{N+1}) = \boxed{S_N + r^{N+1}}$$

$$S_{N+1} = (1 + r + r^2 + \cdots + r^N + r^{N+1})$$

$$= 1 + r(\underbrace{1 + r + \cdots + r^{N-1} + r^N})$$

$$= \boxed{1 + r S_N}$$

So: $\quad S_N + r^{N+1} = 1 + r S_N$

$$S_N - r S_N = 1 - r^{N+1}$$

$$S_N (1 - r) = 1 - r^{N+1}$$

$$\boxed{S_N = \frac{1 - r^{N+1}}{1 - r}}$$

ex $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81} = \sum_{n=0}^{4} r^n$

$$= \frac{1-r^5}{1-r} = \frac{1-(1/3)^5}{1-1/3} = \frac{1-\frac{1}{3^5}}{2/3}$$

$$= \frac{3}{2}\left(1-\frac{1}{3^5}\right)$$

ex $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\frac{2}{81}$

$$= \frac{2}{3}\left[1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}\right] = \frac{2}{3}\left[\underbrace{(1/3)^0+(1/3)^1+(1/3)^2+(1/3)^3}_{\text{formula}}\right]$$

$$= \frac{2}{3}\left(\frac{1-(1/3)^4}{1-1/3}\right) = \cancel{\frac{2}{3}}\left(\frac{1-\frac{1}{3^4}}{\cancel{2/3}}\right)$$

$$= 1-\frac{1}{3^4}$$

Ch. 8.3 Infinite Series

$$\sum_{n=a}^{\infty} a_n = \lim_{N\to\infty} \underbrace{\sum_{n=a}^{N} a_n} = \lim_{N\to\infty} S_N$$

partial sum

ex $\{a_n\}_{n=0}^{\infty} = \{(-1)^n\}_{n=0}^{\infty}$

$1, -1, 1, -1, 1, -1, \ldots$

$S_N = \sum_{k=0}^{N} (-1)^k = 1-1+1-1+1\cdots$

$S_0 = 1$

$S_1 = 0$

$S_2 = 1$

$S_3 = 0$

$\lim_{n\to\infty} S_n$ DNE

So: $\sum_{n=0}^{\infty} a_n$ DIVERGES (limit DNE)

Geometric Series

$$\sum_{k=0}^{N} r^k = \frac{1-r^{N+1}}{1-r} \qquad \text{partial sum}$$

$$\sum_{k=0}^{\infty} r^k = \lim_{N\to\infty}\left(\frac{1-r^{N+1}}{1-r}\right) = \begin{cases} \text{DIVERGE} & \text{if } |r| \geq 1 \\ \frac{1}{1-r} & \text{if } |r|<1 \end{cases}$$

ex $$\underbrace{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k}_{\text{geometric}} = \frac{1}{1-2/3} = \frac{1}{1/3} = 3$$

$$\left|\frac{2}{3}\right| < 1$$

ex $$\underbrace{\sum_{k=2}^{\infty}\left(\frac{2}{3}\right)^k}_{\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots} = \underbrace{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k}_{1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots} - 1 - \frac{2}{3} = 3-1-\frac{2}{3} = 2-\frac{2}{3} = \frac{4}{3}$$

ex $\displaystyle\sum_{n=2}^{\infty} \frac{10^n}{9^{2n}} = \sum_{n=2}^{\infty} \frac{10^n}{81^n} = \sum_{n=2}^{\infty} \left(\frac{10}{81}\right)^n = \sum_{n=0}^{\infty} \left(\frac{10}{81}\right)^n - 1 - \frac{10}{81}$

$\left|\frac{10}{81}\right| < 1$

$$= \left(\frac{1}{1 - 10/81}\right) - 1 - \frac{10}{81} = \frac{1}{71/81} - \frac{91}{81} = \boxed{\frac{81}{71} - \frac{91}{81}}$$

ex $\displaystyle\sum_{n=5}^{\infty} \frac{3^{n+1}}{2^{2n}} = \sum_{n=5}^{\infty} \frac{3 \cdot 3^n}{4^n} = 3 \sum_{n=5}^{\infty} \frac{3^n}{4^n} = 3 \sum_{n=5}^{\infty} \left(\frac{3}{4}\right)^n$

$\left|\frac{3}{4}\right| < 1$

$$= 3\left[\left(\tfrac{3}{4}\right)^5 + \left(\tfrac{3}{4}\right)^6 + \left(\tfrac{3}{4}\right)^7 + \cdots\right]$$

$$= 3 \cdot \left(\tfrac{3}{4}\right)^5 \cdot \left[\underbrace{1 + \left(\tfrac{3}{4}\right) + \left(\tfrac{3}{4}\right)^2 + \cdots}_{\text{formula}}\right]$$

$$= 3\left(\tfrac{3}{4}\right)^5 \underbrace{\sum_{n=0}^{\infty} \left(\tfrac{3}{4}\right)^n} = 3\left(\tfrac{3}{4}\right)^5 \frac{1}{1 - 3/4}$$

$$= \frac{3^6}{4^5}\left(\frac{1}{1/4}\right) = \frac{3^6}{4^4}$$

Usually: it's hard enough to say whether a series converges or diverges.

Note: Suppose $\sum_{n=1}^{\infty} a_n$ converges (say: $\sum_{n=1}^{\infty} a_n = 10$)

That means: $\lim_{N\to\infty} \underbrace{S_N}_{\text{partial sums}} = 10$

So: for really big N, $S_N \approx 10$

So: for really big N, $\underbrace{S_N - S_{N-1}}_{a_N \leftarrow \text{ things I'm adding}} \approx 10 - 10 = 0$

sums

So: If $\sum_{n=1}^{\infty} a_n$ converges,

then $\lim_{n\to\infty} a_n = 0$

ANNOUNCEMENTS

1. Lowest WebWork will be dropped

2. Please fill out course evaluations

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=a}^{\infty} a_n$ DIVERGES

$$\lim_{n\to\infty} a_n$$

$\neq 0$ → $\sum_{n=a}^{\infty} a_n$ DIVERGES

$= 0$ → Maybe $\sum_{n=a}^{\infty} a_n$ converges

Maybe it diverges

We need to think more

We'll learn more tests

ex $\sum_{n=1}^{\infty} \sin n = \sin(1) + \sin(2) + \sin(3) + \cdots$

converge or diverge?

$\lim_{n\to\infty} \sin n$ DNE (jumps around)

So by Divergence Test, $\sum_{n=1}^{\infty} \sin n$ DIVERGES

ex $\sum_{n=1}^{\infty} \frac{n^2+1}{n} = \left(\frac{1+1}{1}\right) + \left(\frac{4+1}{2}\right) + \left(\frac{9+1}{3}\right) + \cdots$

$\lim_{n\to\infty} \left(\frac{n^2+1}{n}\right) = \infty \neq 0$

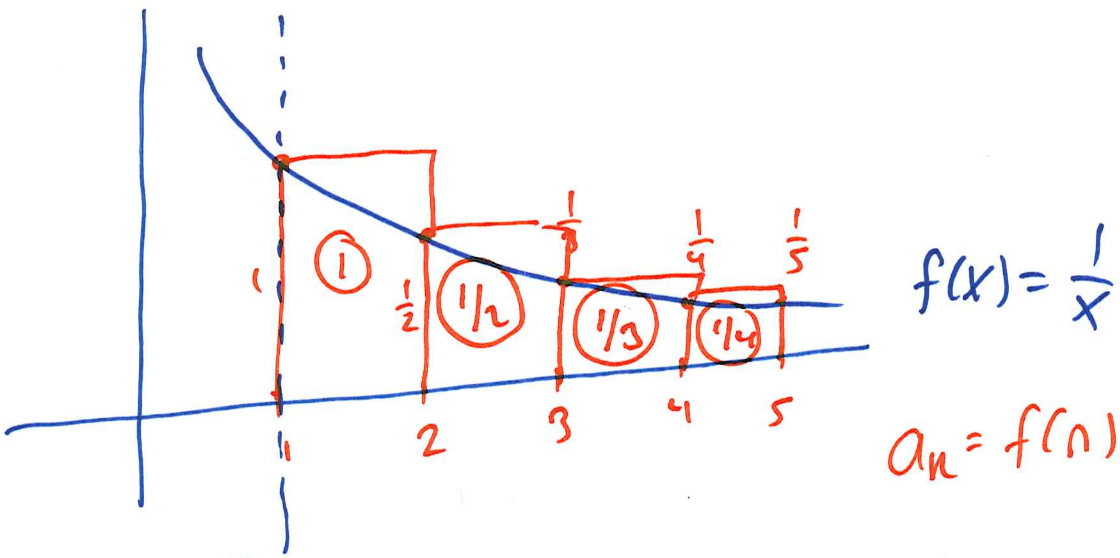
By Divergence Test, $\sum_{n=1}^{\infty} \frac{n^2+1}{n}$ DIVERGES (ex) $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

"Harmonic series"

Let's Try Divergence Test:

$\lim_{n\to\infty} \frac{1}{n} = 0$ No help!

Idea:



Area under rectangles:

$\sum_{n=1}^{\infty} \frac{1}{n}$ $>$

Area under curve $f(x) = \frac{1}{x}$

$\int_1^{\infty} \frac{1}{x}\,dx = \infty$

(p-test)

So $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (DIVERGES)

Integral Test

Let $f(x)$ be a function that is:

- positive
- decreasing
- continuous

on $[a, \infty)$

and let $\{a_n\} = \{f(n)\}$

Then: $\sum_{n=a}^{\infty} a_n$ & $\int_a^{\infty} f(x)\,dx$ either:

- both CONVERGE, or
- both DIVERGE

ex $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

can't use integral test!

$\frac{\cos n}{n^2}$: not all positive

not always decreasing

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ Conv or Div?

Try Divergence Test:

$\displaystyle\lim_{n\to\infty} \frac{1}{n^2+1} = 0$ (no help)

Let $f(x) = \dfrac{1}{x^2+1}$

$f(x)$ is: positive, decreasing, continuous

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \left[\arctan b - \arctan 1\right]$$

$$= \pi/2 - \pi/4 = \pi/4$$

So: $\displaystyle\int_1^{\infty} \frac{1}{x^2+1}\,dx$ CONVERGES

By Integral Test, also

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ CONVERGES (ex) $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n \ln n}$ Converge or Diverge?

Let $f(x) = \dfrac{1}{x \ln x}$

$f(x)$: positive on $[10, \infty)$
decreasing
continuous on $[10, \infty)$

$$\int_{10}^{\infty} \frac{1}{x \ln x}\,dx \underset{\substack{u = \ln x \\ du = \frac{1}{x}dx}}{=} \int_{\ln 10}^{\infty} \frac{1}{u}\,du = \lim_{b \to \infty} \Big[\underbrace{\ln b}_{\to \infty} - \underbrace{\ln(\ln 10)}_{\text{some \#}} \Big]$$

$= \infty$ So integral diverges.

By the Integral Test:

$\displaystyle\sum_{n=10}^{\infty} \frac{1}{n \ln n}$ diverges as well.

ex $\sum_{n=1}^{\infty} \frac{1}{n^p}$, p some positive constant

$f(x) = \frac{1}{x^p}$ ← positive decreasing continuous

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \text{CONVERGING} & p > 1 \\ \text{DIVERGING} & p \le 1 \end{cases}$$

By Integral Test:

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is: $\left.\begin{array}{l} \text{convergent if } p > 1 \\ \text{divergent if } p \le 1 \end{array}\right\}$ p-test for series ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

$p = 2$
converges by p-test

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

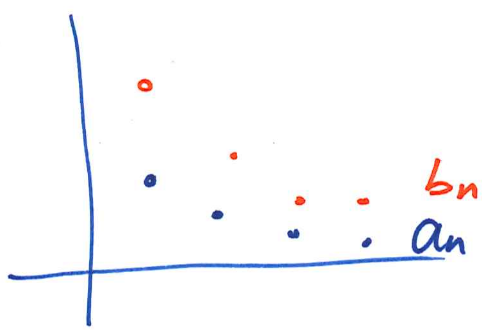
$p = 1/2$
DIVERGES by p-test

Ch 8.5 Ratio & Comparison Tests

Direct Comparison Test:

Let a_n, b_n be sequences with positive terms and $a_n \le b_n$ for all n larger than some constant.

- If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.
- If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges as well.



(ex) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

Does it converge or diverge?

Reasonable guess:

compare to $\displaystyle\sum \frac{1}{n^2}$ ← $p=2$ CONVERGES

Need: $\dfrac{1}{n^2+n} \le \dfrac{1}{n^2}$ (TRUE)

- $\dfrac{1}{n^2+n}$, $\dfrac{1}{n^2}$: positive
- $\dfrac{1}{n^2+n} \le \dfrac{1}{n^2}$
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES by p-test

By Direct Comparison Test,

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ CONVERGES

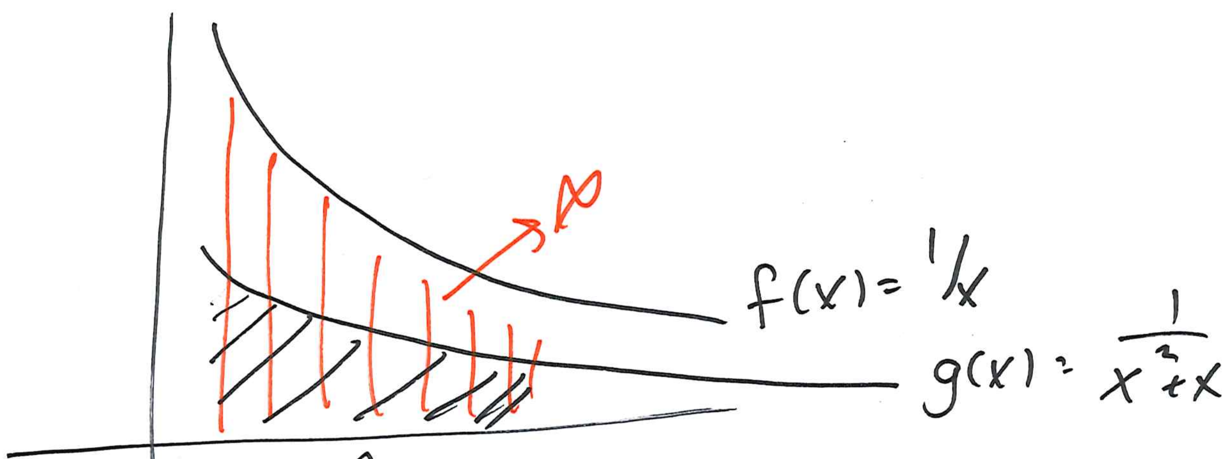
Q: What if we compare $\frac{1}{n^2+n}$ with $\frac{1}{n}$?

$$\sum_{n=1}^{\infty} \frac{1}{n} : \text{DIVERGENT} \quad \text{(Harmonic Series)}$$

$p = 1$

$$\frac{1}{n} \geq \frac{1}{n^2+n}$$

Inequality goes wrong way — can't use direct comparison test



maybe ale ∞
maybe bounded

(ex) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ Conv or div?

Try Div Test:

$\lim_{n\to\infty} \frac{1}{\sqrt{n}-1} = 0$ no help

Think about integral test:

$\int \frac{1}{\sqrt{x}-1}\,dx$ ← unpleasant

+
dec
cont

Notice: $\underbrace{\frac{1}{\sqrt{n}-1}}_{\text{positive}}$ looks a lot like $\underbrace{\frac{1}{\sqrt{n}}}_{\text{positive}}$

$\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} : \quad p = 1/2, \text{ so series DIVERGES}$$

So, by Direct Comparison Test:

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1} \quad \text{DIVERGES}$$

$$\left(\begin{array}{l} \cdot \ \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}-1} \text{ positive} \\ \cdot \ \sum \frac{1}{\sqrt{n}} \text{ DIV} \\ \cdot \ \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n}-1} \end{array} \right)$$

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}-9}$

Want to compare to

$$\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}} = \frac{1}{2}\underbrace{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}}_{\substack{p=1/2 \\ \text{p-test: DIVERGES}}}$$

$n = 100$

$\frac{1}{2\sqrt{n}} = \frac{1}{20} < \frac{1}{2\sqrt{n}-9} = \frac{1}{20-9} = \frac{1}{11}$

In order to use direct comparison test, I need:

$$\underbrace{\frac{1}{2\sqrt{n}}}_{\text{positive}} \leq \underbrace{\frac{1}{2\sqrt{n}-9}}_{\text{positive for } \boxed{\text{large } n}} \quad \text{TRUE}$$

By Direct Comparison Test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}-9}$ also diverges

Limit Comparison Test:

Let a_n, b_n be sequences with positive terms, and

$\lim_{n \to \infty} a_n / b_n$ is a real number (not $\pm\infty$) not 0

Then: $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ either

- both converge, or
- both diverge

What if inequality goes wrong way?

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1+n}{n^3+1}$ conv or div?

Divergence Test: $\displaystyle\lim_{n\to\infty} \frac{1+n}{n^3+1} = 0$ no help

Integral test: partial fractions? ugh

Direct comparison Test:

$$\frac{1+n}{n^3+1} \approx \frac{n}{n^3} = \frac{1}{n^2}$$ compare to this

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ <u>converges</u> ($p=2$, p-test)

<u>NEED</u> (to use direct comparison test)

$$\frac{1+n}{n^3+1} \leq \frac{1}{n^2} \rightarrow n^2+n^3 \leq n^3+1 \quad \text{FALSE}$$

Inequality goes the wrong way!

Compare:

$$\sum \frac{1+n}{n^3+1} \quad \text{and} \quad \sum \frac{1}{n^2}$$

- $\frac{1+n}{n^3+1}$ & $\frac{1}{n^2}$: positive

- $\lim\limits_{n\to\infty} \left(\frac{1+n}{n^3+1}\right) \Big/ \left(\frac{1}{n^2}\right) = \lim\limits_{n\to\infty} \frac{n^2(1+n)}{n^3+1}$

$= \lim\limits_{n\to\infty} \frac{n^3+n}{n^3+1} = \frac{1}{1} = 1 \neq 0$

- $\sum \frac{1}{n^2}$ converges ($p=2$, p-test)

By Limit Comparison Test, $\sum \frac{1+n}{n^3+1}$ also $\boxed{\text{converges}}$

(ex) $\sum_{n=1}^{\infty} \frac{n^2+n+5}{n^4+3n+6}$ conv or div ?

Compare : $\frac{n^2}{n^4} = \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONV, $p=2$, p-test

For Direct Comparison Test:

need $\frac{n^2+n+5}{n^4+3n+6} \leq \frac{1}{n^2}$ ← I'm lazy & I don't want to check this!

Limit Comparison Test :

$$\lim_{n\to\infty} \left(\frac{n^2+n+5}{n^4+3n+6}\right) \Big/ \left(\frac{1}{n^2}\right) = \lim_{n\to\infty} \frac{n^4+n^3+5n^2}{n^4+3n+6} = \frac{1}{1} = 1 \neq 0$$

- Both series have positive terms

- $\lim_{n \to \infty} (\quad)/(1/n^2) = 1 \neq 0$

- $\sum_{n=1}^{\infty} \frac{1}{n^2}$ <u>converges</u> by p-test

So: by Limit Comparison Test,

$\sum_{n=1}^{\infty} \frac{n^2+n+5}{n^4+3n+6}$ also <u>converges</u>

ex) $\sum 3(1.001)^k$

Geometric, $r=1.001>1$
So DIVERGES
(can also use div test)

ex) $\sum \underbrace{\frac{n^2+1}{n+2}}_{\text{terms: } \to \infty}$

DIVERGES
(by Divergence Test)

ex) $\sum \frac{1}{e^n} = \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$

$= \frac{1}{1-1/e}$

Geometric,
$r = \frac{1}{e} < 1$
So: converges

→ Ch 9.1 → optional

Taylor Polynomials; Taylor Remainder Theorem

→ Quiz 6: Take-home

(please staple; name & SID in upper-right corner)

→ Please fill out course evaluations online

esp comment on quizzes

So far, we've learned these tests:

- Divergence Test
- Integral Test
- Direct Comparison Test
- Limit Comparison Test
- p-test

only work on series that have positive terms

Absolute Convergence Theorem!

If $\sum_{n=a}^{\infty} |a_n|$ converges, then $\sum_{n=a}^{\infty} a_n$ converges, too.

e.g. $\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}}$ Conv or Div?

$-\frac{1}{1} + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} \cdots$

terms not all positive

Consider a different series:

$$\left.\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}\right\}$$ converges by p-test ($p=2$)

$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$

So, by Absolute Convergence Theorem, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges (absolutely).

If $\sum a_n$ <u>converges</u>, there are two possibilities:

1. $\sum |a_n|$ converges — We say $\sum a_n$ <u>converges absolutely</u>

2. $\sum |a_n|$ diverges — We say $\sum a_n$ <u>converges conditionally</u>

(ex) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^3+1}$ Conv or Div?

Some negative terms

Consider:

$$\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^3+1}\right|$$

Note:

- $\left|\frac{\cos n}{n^3+1}\right| \le \frac{1}{n^3}$
- Both sequences have positive terms
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges ($p=3$ p-test)

So, by Direct Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^3+1}\right|$ converges.

By Absolute Convergence Theorem, $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^3+1}$ converges. (absolutely)

Ratio Test

Idea: compare mystery series to a geometric series

Let $\sum_{n=a}^{\infty} a_n$ be a series with positive terms,
and let $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

① If $0 \le r < 1$, the series converges
② If $r > 1$ (including $r = \infty$), the series diverges
③ If $r = 1$: need another test

ex $\sum_{n=1}^{\infty} \underbrace{\frac{n^2}{4^n}}_{\text{positive}}$ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{(n+1)^2}{4^{n+1}}\right) \Big/ \left(\frac{n^2}{4^n}\right)$

$= \lim_{n\to\infty} \frac{(n+1)^2}{4^{n+1}} \cdot \frac{4^n}{n^2} = \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n}\right)^2}_{1^2} \cdot \frac{\cancel{4^n}}{\cancel{4^n} \cdot 4} = \frac{1}{4} < 1$

So $\sum_{n=1}^{\infty} \frac{n^2}{4^n}$ converges by ratio test.

(ex) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$ Use Ratio Test

- Positive Terms

- $r = \displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{(n+1)^5} \Big/ {}^{1}/_{n^5}$

$= \displaystyle\lim_{n\to\infty} \underbrace{\frac{n^5}{(n+1)^5}}_{\underbrace{\left(\frac{n}{n+1}\right)^5}_{1^5 = 1}} = \frac{1}{1} = 1$

Ratio Test: inconclusive ($r=1$)

Use p-test: $p = 5 > 1$, so $\sum \frac{1}{n^5}$ <u>converges</u>.

Quick Review: factorials

$$n! = n(n-1)(n-2)\cdots(1)$$

e.g. $4! = 4\cdot 3\cdot 2\cdot 1 = 24$

$5! = 5\cdot\underbrace{4\cdot 3\cdot 2\cdot 1} = 5\cdot 4! = 5\cdot 24 = 120$

$$\frac{5!}{4!} = \frac{5\cdot\cancel{4\cdot 3\cdot 2\cdot 1}}{\cancel{4\cdot 3\cdot 2\cdot 1}} = 5$$

Similarly: $\dfrac{(n+1)!}{n!} = \dfrac{(n+1)\,\cancel{n(n-1)(n-2)\cdots(1)}}{\cancel{n(n-1)(n-2)\cdots(1)}} = n+1$

$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{n!}}_{a_n}$$

Conv or Div?

Ratio Test:

- positive terms

- $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{1}{(n+1)!}\right) \Big/ \frac{1}{n!}$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

So, by Ratio Test, $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges ex $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{(-2)^n}$ Conv or Div?

$2^{n-1} = 2^n \cdot 2^{-1} = \dfrac{2^n}{2}$

Can only use Ratio Test if positive terms

$$\sum_{n=1}^{\infty} \left| \frac{n^5}{(-2)^n} \right| = \sum_{n=1}^{\infty} \frac{n^5}{2^n}$$ } - positive terms - use ratio test

$2^{n+1} = 2^1 2^n = 2 \cdot 2^n$

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^5}{2^{n+1}} \Big/ \frac{n^5}{2^n}$$

$$\frac{(n+1)^5}{n^5} = \left(\frac{n+1}{n}\right)^5$$

$$\frac{x^2}{y^2} = \frac{x \cdot x}{y \cdot y} = \frac{x}{y} \cdot \frac{x}{y} = \left(\frac{x}{y}\right)^2$$

$$= \lim_{n\to\infty} \frac{(n+1)^5}{2^{n+1}} \cdot \frac{2^n}{n^5}$$

$$= \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n}\right)^5}_{1^5 = 1} \cdot \frac{\cancel{2^n}}{\cancel{2^n} \cdot 2} = \frac{1}{2} < 1$$

So $\sum \left| \frac{n^5}{(-2)^n} \right|$ converges by Ratio Test

By Absolute Convergence Theorem:

$\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{(-2)^n}$ converges (absolutely)

Quick Review: Taylor Polynomials

Centre (a)

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$

$$\boxed{T_3(a) = f(a)} + 0 + 0 + 0$$

$$T_3'(x) = f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2}(x-a)^2$$

$$\boxed{T_3'(a) = f'(a)} + 0 + 0$$

$$T_3''(x) = f''(a) + f'''(a)(x-a)$$

$$\boxed{T_3''(a) = f''(a)} + 0$$

$$T_3'''(x) = f'''(a)$$

$$\boxed{T_3'''(a) = f'''(a)}$$

$$T_3^{(4)}(x) = 0$$

$$T_3^{(5)}(x) = 0$$

$$\vdots$$

Taylor Poly:

$$\sum_{n=1}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Series:

$$\sum_{n=1}^{\infty} \underbrace{\frac{f^{(n)}(a)}{n!}}_{\substack{\text{constants:} \\ \text{they change with } n \\ \text{no } x}} (x-a)^n$$

variable

constant

Power Series:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

$\{c_n\}$: sequence of constants (no x)

x : variable

a : constant "centre"

Vocab:

The set of values of x for which it converges: Interval of Convergence, I

The radius of convergence, R, is the distance from the centre to the boundary of the interval of convergence.

ex $\sum_{n=0}^{\infty} x^n$ Power Series (Geometric)

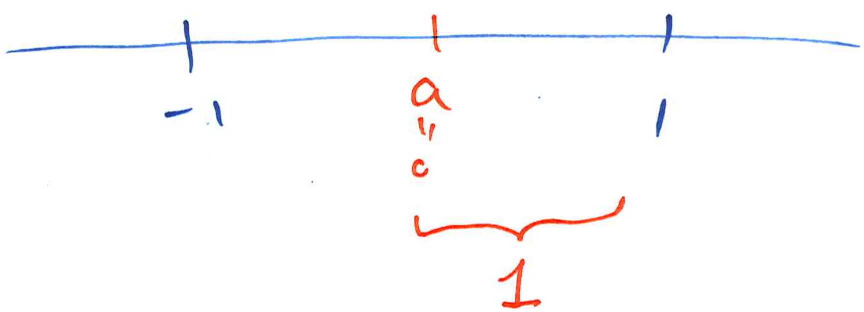
Converges when $|x|<1$ i.e. $-1<x<1$

$\boxed{(-1, 1)}$ "Interval of Convergence"

Centre: $a=0$

Radius of Convergence:

$R=1$



p694 : big list

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$R = \infty$

i.e.

Int. of Convergence

$(-\infty, \infty)$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$R = \infty$

$a = 0$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$R = \infty$

$a = 0$

ex $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ For which values of x does it converge?

$\sum \left|\frac{x^k}{k!}\right|$: $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \left|\frac{x^{k+1}}{(k+1)!}\right| \Big/ \left|\frac{x^k}{k!}\right|$

$= \lim_{k\to\infty} \frac{|x^{k+1}|}{(k+1)!} \cdot \frac{k!}{|x^k|}$

$= \lim_{k\to\infty} \frac{|x|^{k+1}}{|x|^k} \cdot \frac{k!}{(k+1)!}$

$= \lim_{k\to\infty} \frac{|x| \cdot \cancel{|x|^k}}{\cancel{|x|^k}} \cdot \frac{\cancel{k!}}{(k+1)\cancel{k!}}$

$= \lim_{k\to\infty} \frac{|x|}{(k+1)} = 0$ no matter what constant we plug in for x

So by Ratio Test:

$\sum \frac{x^k}{k!}$ converges for all x.

ex $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{3^k}$ For which values of x does this converge?

Try to write as $\sum r^k$ (geometric)

$$\sum_{k=0}^{\infty} \frac{x \cdot x^{2k}}{3^k} = x \sum_{k=0}^{\infty} \frac{x^{2k}}{3^k} = x \sum_{k=0}^{\infty} \frac{(x^2)^k}{3^k}$$

$$= x \sum_{k=0}^{\infty} \left(\frac{x^2}{3}\right)^k$$

Geom series, $r = \frac{x^2}{3}$

Conv. when $|r| < 1$

ie $-1 < r < 1$

$$-1 < \frac{x^2}{3} < 1$$

$$-3 < x^2 < 3$$

$$|x| < \sqrt{3}$$

So: $-\sqrt{3} < x < \sqrt{3}$

Interval of Convergence:
$(-\sqrt{3}, \sqrt{3})$

Radius of Convergence:
$\sqrt{3}$

ex $\sum_{k=0}^{\infty} k!\,(x-2)^k$

Which values of x make it converge?

Factorials $\to$ Ratio

$$r = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \left| \frac{(k+1)!\,(x-2)^{k+1}}{k!\,(x-2)^k} \right| = \lim_{k\to\infty} (k+1)\,|x-2|$$

cancellations

If $x=2$: $r=0$, so $\sum k!(x-2)^k$ converges

If $x\neq 2$: $r=\infty$ so $\sum k!(x-2)^k$ diverges

Interval of Convergence: $[2,2]$ $(x=2)$

Radius of Convergence: $R=0$

$a=2$

centre

Telescoping Series

(lots of cancellations)

ex $$\sum_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n+3}\right) = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(\frac{1}{3}-\frac{1}{N+3}\right) = \frac{1}{3}$$

$n=1$ $\quad \frac{1}{3} - \cancel{\frac{1}{4}}$

$n=2$ $\quad + \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}$

$n=3$ $\quad + \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}$

$n=4$ $\quad + \cancel{\frac{1}{6}} - \cancel{\frac{1}{7}}$

$n=5$ $\quad + \cancel{\frac{1}{7}} - \frac{1}{8}$

$\vdots$

$S_1 = \frac{1}{3} - \frac{1}{4}$

$S_2 = \frac{1}{3} - \frac{1}{5}$

$S_3 = \frac{1}{3} - \frac{1}{6}$

$S_5 = \frac{1}{3} - \frac{1}{8}$

$S_N = \frac{1}{3} - \frac{1}{N+3}$

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$

partial fractions

$\left(\text{Conv: } \frac{1}{n(n+1)} < \frac{1}{n^2} \right)$ $\quad \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$

$n=1 \quad \frac{1}{1} - \cancel{\frac{1}{2}}$

$n=2 \quad + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$

$n=3 \quad + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}$

$n=4 \quad + \cancel{\frac{1}{4}} - \frac{1}{5}$

$\vdots$

$S_N = 1 - \frac{1}{N+1}$

$\lim_{N\to\infty} S_N = 1 - 0 = \boxed{1}$

ex $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \left[\ln(n+1) - \ln n\right]$

DIVERGES

$$
\begin{array}{l|l}
n=1 & \ln 2 - \cancel{\ln 1}^{\,0} \\
n=2 & + \ln 3 - \ln 2 \\
n=3 & + \ln 4 - \ln 3 \\
n=4 & + \ln 5 - \ln 4 \\
 & \quad \vdots
\end{array}
$$

$s_4 = \ln 5$

$s_N = \ln(N+1)$

$\displaystyle\lim_{N\to\infty} s_N = \infty$

Table 9.5, p694 has examples of Taylor Series

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$$

Interval of Convergence: $(-\infty, \infty)$

Radius of Convergence: ∞

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$$

(ex) $\sum_{n=0}^{\infty} n!\,(x-2)^n$

For which values of x does it converge?

Consideration: If $x<2$, terms not all positive.

Consideration for divergence test:

$$\lim_{n\to\infty} n!\,(x-2)^n$$

FACT: diverges whenever $x \neq 2$

↑

not obvious

If $x = 2.5$:

$$\lim_{n\to\infty} n!\,(2.5-2)^n$$

$$= \lim_{n\to\infty} \underbrace{n!}_{\text{bigger}} \underbrace{\left(\tfrac{1}{2}\right)^n}_{\text{smaller}}$$

unclear what limit is when x is close to 2

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!\,\cancel{(x-2)}^{n+1}}{n!\,\cancel{(x-2)^n}} = \lim_{n\to\infty} (n+1)\underbrace{(x-2)}_{\text{fixed number}} = \pm\infty$$
if $x \neq 2$

$a_1 + a_2 + a_3 + a_4 + a_5 \ldots$

$\frac{a_2}{a_1}$ $\frac{a_3}{a_2}$

When n is big, I have to multiply a_n by a huge # to get a_{n+1}

i.e. $|a_n|$ is growing hugely

So: If $x \neq 2$, $\lim_{n\to\infty} a_n$ DNE

by DIVERGENCE TEST, $\sum n!(x-2)^n$ DIV when $x \neq 2$.

Ch 9.2 (cont'd) Manipulating Power Series

Thm 9.4 p 679
(paraphrase)

- addition + subtraction "work"

e.g. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$ for all x

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$ for all x

$[\sin x + \cos x] = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} \cdots$ for all x

- Multiplication by an appropriate power of x
  also "works"

ex) Find a power series that converges to: $\dfrac{x^3}{1-x}$

$$x^3\Bigg(\underbrace{\frac{1}{1-x}}_{\frac{1}{1-r}}\Bigg) = x^3 \underbrace{\sum_{n=0}^{\infty} x^n}_{\sum r^n} = \sum_{n=0}^{\infty} x^3 \cdot x^n = \sum_{n=0}^{\infty} x^{n+3}$$

conv when $|x|<1$

$$= \sum_{n=3}^{\infty} x^n$$

conv when $|x|<1$

ex) $\underbrace{\frac{1}{x}}_{x^{-1}\sin x}\sin x$ vs $\frac{1}{x}\cos x$

$$\frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\right)$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

Still a power series — o/c

$$\frac{1}{x}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\right)$$

$$= \underbrace{\frac{1}{x}} - \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} \cdots$$

not a power series

- Composition $f(g(x))$ is ok

if $g(x) = b x^m$, appropriate m

const

e.g.

$$\frac{1}{1-(7x^2)} = \sum_{n=0}^{\infty} (7x^2)^n = \underbrace{\sum_{n=0}^{\infty} 7^n x^{2n}}$$

- still a power series
- converge to $\frac{1}{1-7x^2}$ (when it converges)

conv when

$|7x^2| < 1$

$7x^2 < 1$

$x^2 < \frac{1}{7}$

$|x| < \frac{1}{\sqrt{7}}$

$\boxed{-\frac{1}{\sqrt{7}} < x < \frac{1}{\sqrt{7}}}$ Interval of Convergence

Also: Integration & Differentiation of Power Series "works"

(ex) $\int_0^1 e^{x^2}\,dx$

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$ (table 9.5)

$$e^{(x^2)} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + \frac{x^2}{1} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

$n=0$: $\frac{x^0}{0!} = \frac{1}{1} = 1$

convention

$$\int e^{x^2}\,dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!\,(2n+1)} = x + \frac{x^3}{1\cdot 3} + \frac{x^5}{2!\cdot 5} + \frac{x^7}{3!\,7}$$

$$\int_0^1 e^{x^2}\,dx = F(1) - F(0) = \sum_{n=0}^{\infty} \frac{1}{n!(2n+1)} - \cancel{\sum_{n=0}^{\infty} 0} = \sum_{n=0}^{\infty} \frac{1}{n!(2n+1)}$$

$$= \frac{1}{1} + \frac{1}{3} + \frac{1}{2(5)} + \frac{1}{6(10)} + \cdots$$

← add first several terms to get approximation

$$\sum_{n=0}^{\infty} \frac{1}{n!\,(2n+1)} = \lim_{N\to\infty} \underbrace{\sum_{n=0}^{N} \frac{1}{n!\,(2n+1)}}_{=\,?}$$

(ex) Find a power series that converges to $\arctan x$.

Notice: $\arctan x = \int \underbrace{\frac{1}{1+x^2}}_{\text{close to } \frac{1}{1-x}} dx + C$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$\frac{1}{1-r}$ $(r=-x^2)$ $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ when $|r|<1$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Interval of Conv:

$|-x^2|<1$

$x^2<1$

$\boxed{-1<x<1}$

$$\arctan x + C = \int \frac{1}{1+x^2}\,dx = \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Last Step: check $+C$

Plan: plug in $x=0$

Know: $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$

If $x=0$: $\arctan 0 = \underbrace{\left(\sum_{n=0}^{\infty} \underbrace{(-1)^n \frac{0^{2n+1}}{2n+1}}_{0}\right)}_{0} + C$

$\arctan 0 = C$

$0 = C$

So: $\boxed{\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$

will converge when

$-1 < x < 1$

(ex) Find a power series converging to:

$\ln|x-1|$

Note: $\int \frac{1}{x-1}\,dx = \ln|x-1| + C$

$$\int \underbrace{\frac{1}{1-x}}_{\frac{1}{1-r}}\,dx = \int \frac{-1}{x-1}\,dx$$

So: $\ln|x-1| = -\int \underbrace{\frac{1}{1-x}}_{\sum x^n}\,dx + C$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

So $$\underbrace{\int \frac{1}{1-x}\,dx}_{= -\ln|x-1|+C} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$\ln|x-1| = -\sum_{n=1}^{\infty} \frac{x^n}{n} + C$$

$$= \left(\sum_{n=1}^{\infty} -\frac{x^n}{n}\right) + C$$

So: $$\boxed{\ln|x-1| = \sum_{n=1}^{\infty} -\frac{x^n}{n}}$$

To find C, set $x=0$:

$$\ln|0-1| = \left(\sum_{n=1}^{\infty} -\frac{0^n}{n}\right) + c$$

$$0$$

$$0 = C$$

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\cdot 3^n}$

If $x = \frac{1}{3}$:

$$\ln\left|\tfrac{1}{3}-1\right| = \sum_{n=1}^{\infty} -\frac{(1/3)^n}{n} = \sum_{n=1}^{\infty} -\frac{1}{n\cdot 3^n}$$

$$-\ln\left|\tfrac{1}{3}-1\right| = \sum_{n=1}^{\infty} \frac{1}{n\cdot 3^n}$$

$\parallel$

$$-\ln\left|\tfrac{-2}{3}\right| = -\ln\left|2/3\right| = \boxed{\ln(3/2)} = \ln 3 - \ln 2$$

Ch 9.3 Taylor Series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

Taylor Series for $f(x)$, centred at a

If $a=0$, we also call it a Maclaurin Series

(ex) Find Taylor Series for $f(x) = \ln x$, centred at $x=1$.

$f(x) = \ln x$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = -x^{-2}$

$f'''(x) = (-1)(-2)x^{-3}$

$f^{(4)}(x) = (-1)(-2)(-3)x^{-4}$

$f^{(5)}(x) = (-1)(-2)(-3)(-4)x^{-5}$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,x^{-n}$

where does pattern start? at $n=1$

If $n=1$:

$f'(x) = (-1)^0\, 0!\, x^{-1}$

$= 1 \cdot 1 \cdot x^{-1}$ ✓

if $n=0$, $f^{(n)}(a) = f(1) = \ln(1) = \underline{\underline{0}}$

If $n \geq 1$, $\underline{\underline{f^{(n)}(a)}} = f^{(n)}(1) = \underline{\underline{(-1)^{n-1}}}(n-1)!(1)$

Taylor series: $$\sum_{n=0}^{\infty} \frac{\boxed{f^{(n)}(a)}}{n!}(x-a)^n$$

$$= 0 + \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n = \ln x$$ if it converges

(ex) Use Taylor Series we just found to approximate $\ln(9/10)$.

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n \qquad (\text{if it converges}) \leftarrow \text{depends on } x$$

$x = 9/10$

So: $$\ln(9/10) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{9}{10}-1\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{-1}{10}\right)^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-1)^n}{n \cdot 10^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n \cdot 10^n} = \underbrace{\sum_{n=1}^{\infty} \frac{-1}{n \cdot 10^n}}_{\text{conv or div?}}$$

$$= -\underbrace{\sum_{n=1}^{\infty} \frac{1}{n \cdot 10^n}}_{\text{consider this series (positive terms)}}$$

Notice: $\dfrac{1}{n \cdot 10^n} < \dfrac{1}{10^n}$ positive terms

$$\sum_{n=1}^{\infty} \frac{1}{10^n} = \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n}_{\text{geometric}} = \frac{1}{1-1/10} \quad \text{converges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n\cdot 10^n}$$ converges by Direct Comparison Test

So $$-\sum_{n=1}^{\infty} \frac{1}{n\cdot 10^n}$$ converges too

$$\sum_{n=1}^{\infty} \frac{-1}{n\cdot 10^n} = \ln(9/10)$$

$$\frac{-1}{1\cdot 10^1} + \frac{-1}{2\cdot 10^2} + \frac{-1}{3\cdot 10^3} + \frac{-1}{4\cdot 10^4} + \cdots = \ln(9/10)$$

Approx: $\frac{-1}{10} = -0.1$

$$\frac{-1}{200} = \frac{-0.5}{100} = -0.005$$

$$-0.105 \approx \ln(9/10)$$

(ex)

Find Taylor Series of $f(x) = e^{2x}$, centred at $x = \frac{1}{2}\ln 2$

Definition: $$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(x) = e^{2x} \longrightarrow f(\tfrac{1}{2}\ln 2) = e^{2\cdot\frac{1}{2}\ln 2} = e^{\ln 2} = 2^1$$

$$f'(x) = 2e^{2x} \longrightarrow f'(\tfrac{1}{2}\ln 2) = 2\cdot e^{2\cdot\frac{1}{2}\ln 2} = 2^2$$

$$f''(x) = 2^2 e^{2x} \longrightarrow f''(\tfrac{1}{2}\ln 2) = 2^2 e^{2\cdot\frac{1}{2}\ln 2} = 2^3$$

$$f'''(x) = 2^3 e^{2x} \longrightarrow f'''(a) = 2^4$$

In general: $f^{(n)}(\frac{1}{2}\ln 2) = 2^{n+1}$ all $n \geq 0$

Taylor Series: $$\boxed{\sum_{n=0}^{\infty} \frac{2^{n+1}}{n!}\left(x - \tfrac{1}{2}\ln 2\right)^n}$$

Ch 9.4 Working with Taylor Series

(compare to 9.2)

(ex) Show that $\frac{d}{dx}\{\sin x\} = \cos x$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{-x^6}{6!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\frac{d}{dx}\{\sin x\} = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n+1)!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \underline{\cos x}$$

(ex) Approximate e.

Note: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, all x

$\boxed{x=1}$

$e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}$

Fact: $e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

Approx: add up first several terms

$\underbrace{\frac{1}{1} + \frac{1}{1} + \frac{1}{2}}_{2.5} + \underbrace{\frac{1}{6}}_{0.1\overline{6}} + \frac{1}{24}$

$\frac{1}{24} \approx \frac{1}{25} = \frac{4}{100} = 0.04$

$$\begin{array}{r} 2.5 \\ +\ 0.1\overline{6} \\ \hline \boxed{2.\overline{6}} \end{array}$$ approx (actually $e \approx 2.718\ldots$)

ex $\displaystyle\sum_{k=1}^{\infty} \frac{(0.1)^k}{k}$

Vague Question:
What's going on w/ this series?

① Does it converge? → Yes (already saw $\sum \frac{(1/10)^k}{k}$

② To what?

From Table:

$$-\ln(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k}, \quad \text{if } -1 \le x < 1$$

use $x = 0.1$

$$-\ln(1-0.1) = \underline{\underline{\sum_{k=1}^{\infty} \frac{(0.1)^k}{k}}}$$

$$\| $$

$$-\ln(9/10) = \boxed{\ln(10/9)}$$

ex $\displaystyle\sum_{k=0}^{\infty} \frac{2^k 3^{k+10}}{k!} = ?$

$$\| $$

$$\sum_{k=0}^{\infty} \frac{2^k \cdot 3^k \cdot 3^{10}}{k!} = 3^{10} \sum_{k=0}^{\infty} \frac{6^k}{k!} = \boxed{3^{10} \cdot e^6}$$

Note: $\displaystyle e^x = \sum_{n=0}^{\infty} \frac{x^k}{k!}$ $\quad x=6$:

$$\sum_{k=0}^{\infty} \frac{6^k}{k!} = e^6$$

ex Evaluate $\displaystyle\sum_{n=0}^{\infty} n(n-1)\frac{1}{2^{n-2}}$

Start with:

$$(1-x)^{-1} = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$ when $|x|<1$

} differentiate

$$+(1-x)^{-2} = \sum_{n=0}^{\infty} n x^{n-1}$$

} again

$$+2(1-x)^{-3} = \sum_{n=0}^{\infty} n(n-1)x^{n-2}$$

use $x=\frac{1}{2}$

$$2(1-\tfrac{1}{2})^{-3} = \sum_{n=0}^{\infty} n(n-1)\left(\tfrac{1}{2}\right)^{n-2}$$

$$= 2(\tfrac{1}{2})^{-3} = \sum_{n=0}^{\infty} n(n-1)\cdot\frac{1}{2^{n-2}}$$ } STARTED WITH

$$= 2\cdot 2^3 = \boxed{16}$$

ex Recall:

$$2 - x^2 = ax^2 + bx + c$$

What are a, b, c:

$c = 2$

$a = -1$

$b = 0$

Can also do w/ Taylor Series

(ex) Suppose $f(x)$ has Taylor Series (Maclaurin)

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!(2k)!}$$

$\leftarrow$ centre

What are derivatives of f at $x=0$

(k=11)

$$f(x) = \frac{x^0}{0!(0)!} + \frac{x^1}{1!2!} + \frac{x^2}{2!4!} + \frac{x^3}{3!6!} \cdots$$

$\frac{1}{1}$ $\quad$ $\frac{1}{2}x$

$$= f(0) + f'(0)\cdot x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 \cdots$$

$\frac{x^{11}}{11!\,22!}$ (given)

$\frac{f^{(11)}(0)}{11!}x^{11}$ (def of Taylor)

$f(0)=1$
(constant term)

$f'(0) = \frac{1}{1\cdot 2} = \frac{1}{2}$

$\frac{f'''(0)}{\cancel{3!}} = \frac{1}{\cancel{3!}6!}$

So $f'''(0) = \frac{1}{6!}$

$$\boxed{f^{(11)}(0) = \frac{1}{22!}}$$

ex Taylor Series of $f(x)$ is:

$$f(x) = \sum_{k=1}^{\infty} \frac{x^{2k}}{k} = \frac{x^2}{1} + \frac{x^4}{2} + \frac{x^6}{3} + \frac{x^8}{4} + \cdots$$

$\frac{x^{12}}{6}$ $(k=6)$

Q: What is $f^{(11)}(0)$? $\boxed{=0}$

All odd derivs @ $x=0$ are 0

Q: What is $f^{(12)}(0)$?

$$\frac{1}{6} = \frac{f^{(12)}(0)}{12!}$$

So $\boxed{f^{(12)}(0) = \frac{12!}{6}}$

Taylor Series: (Maclaurin) $f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$

$$\frac{f^{(12)}(0)}{12!}x^{12}$$