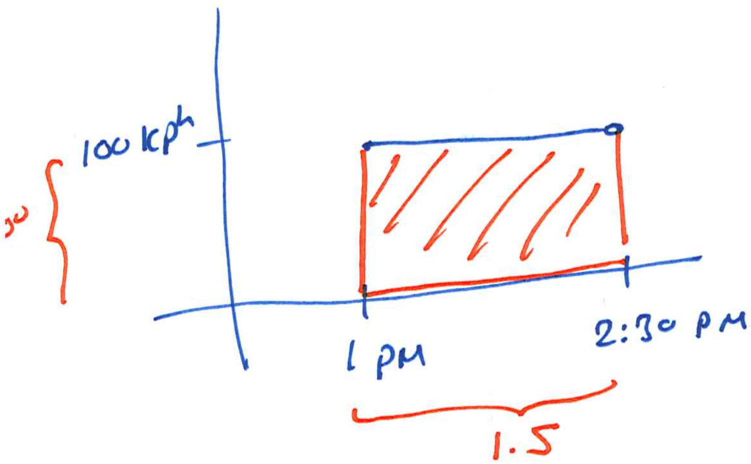
Ch. 5.1 : Approximating Area under Curves

DISTANCE = RATE × TIME

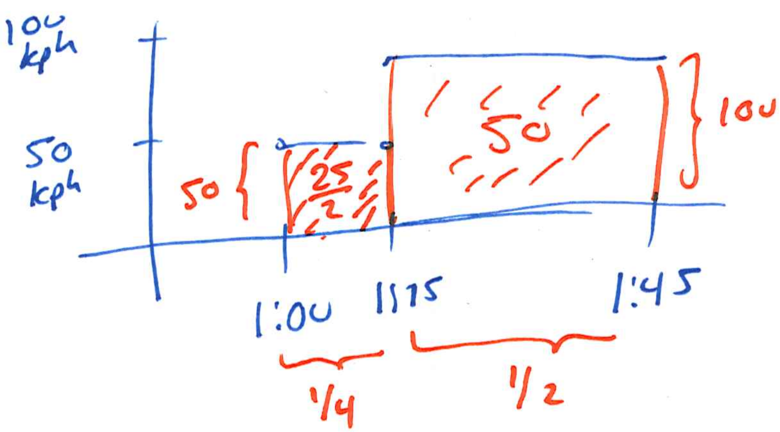


Dist travelled from 1 pm to 2:30 pm:

$(100 \frac{km}{hr})(1.5\ hr)$

$= \boxed{150\ km}$

Area of rectangle under line



Dist travelled from 1 – 1:45 :

1–1:15 : $\frac{1}{4}$ hr, 50 kph

dist : $\frac{50}{4} = \frac{25}{2}$ km

1:15–1:45 : $\frac{1}{2}$ hr, 100 kph

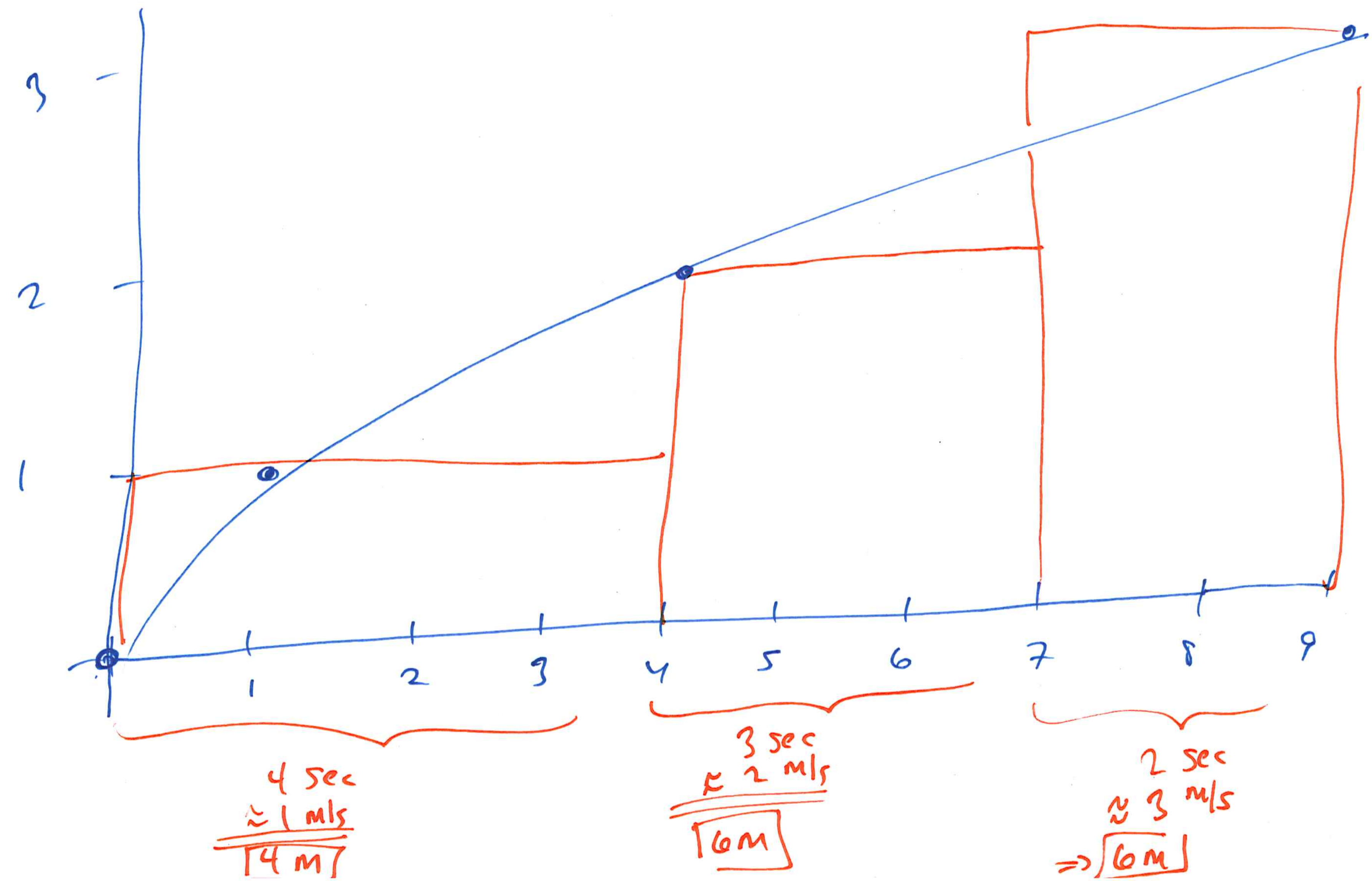
dist : $(\frac{1}{2})(100) = 50$ km

All together : $\boxed{50 + \frac{25}{2}\ km}$

DISTANCE TRAVELLED: AREA UNDER LINE

At time t, speed: $s(t) = \sqrt{t}$ m/s

APPROX DIST TRAVELLED ≈ 16 m

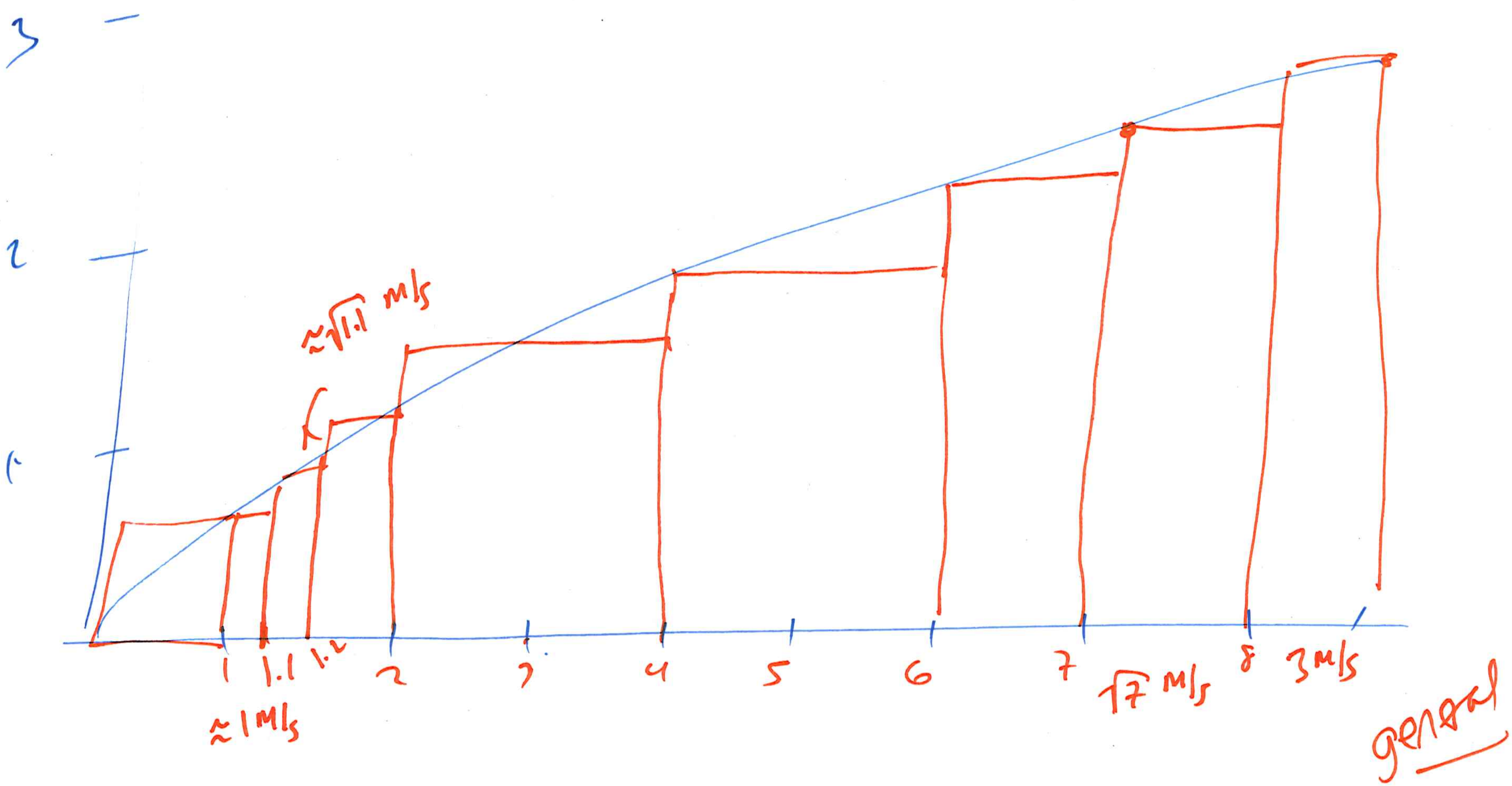


Grid pointss 0, 1, 1.1, 1.2, 2, 4, 6, 7, 8, 9

Partition: Regular if cuts all same size

width of intervals: Δx

General if not same size

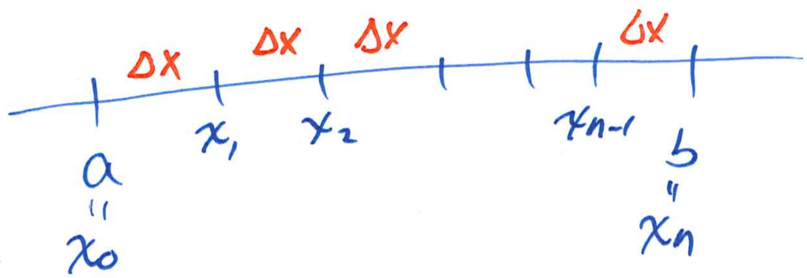


In a regular partition from a to b, with n intervals:

$$\Delta x = \frac{b-a}{n}$$

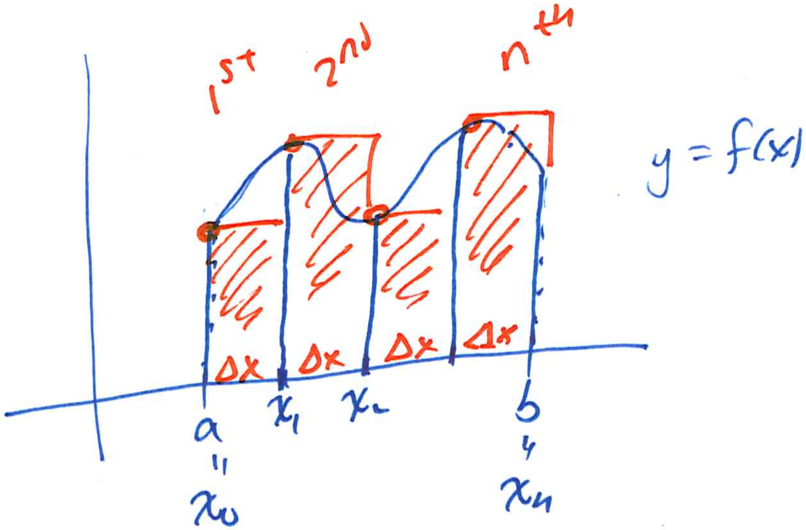
Grid points:

$$x_i = a + i\Delta x$$



Riemam Sums

Regular partitions



Left Riemann Sum

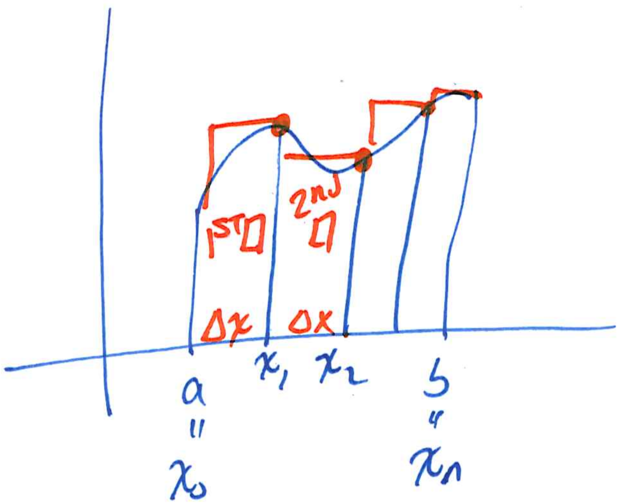
Area of i^{th} rectangle:

$= (\text{base})(\text{height})$

$= \Delta x \cdot f(x_{i-1})$

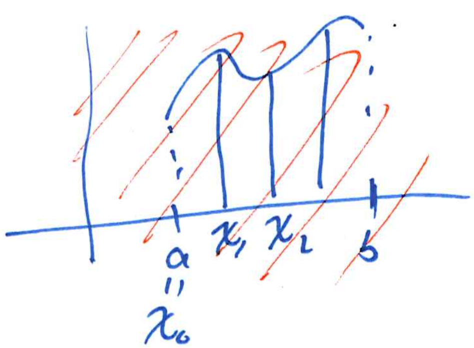
Approx of ttl area:

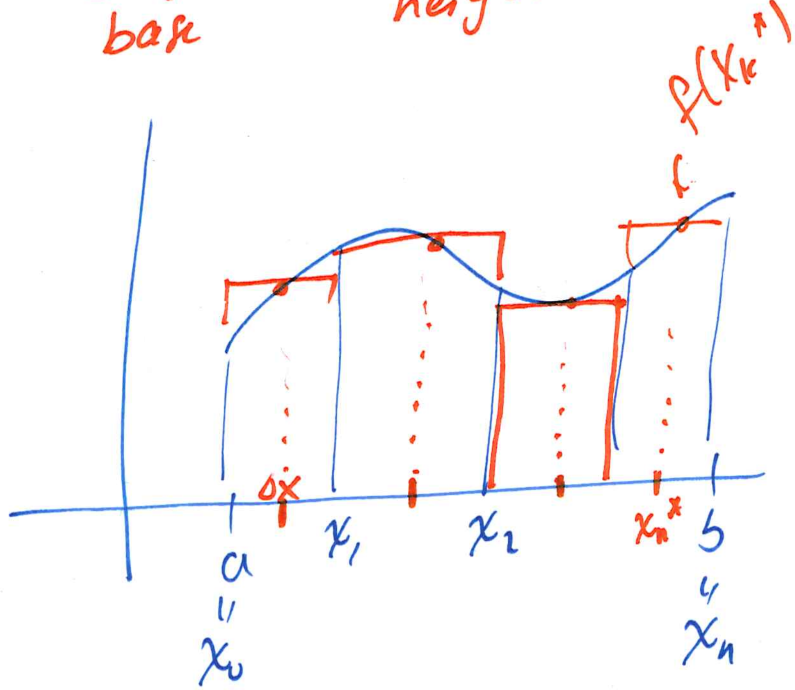
$\Delta x f(x_0) + \Delta x\, f(x_1) + \Delta x\, f(x_2) + \dots + \Delta x\, f(x_{n-1})$



Right Riemann Sum

Area of i^{th} □ : $\underbrace{(\Delta x)}_{\text{base}} \cdot \Big(\underbrace{f(x_i)}_{\text{height}} \Big)$





Midpoint Riemann Sum

height of i^{th} rectangle:

$$f\left(\frac{x_{i-1}+x_i}{2}\right)$$

base: ΔX

Riemann Sum

Suppose f is defined on a closed interval $[a,b]$, which is divided into n intervals of equal length, Δx.

If x_k^* is any point in the k^{th} interval $[x_{k-1}, x_k]$, for $k=1,\dots,n$, then

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x$$

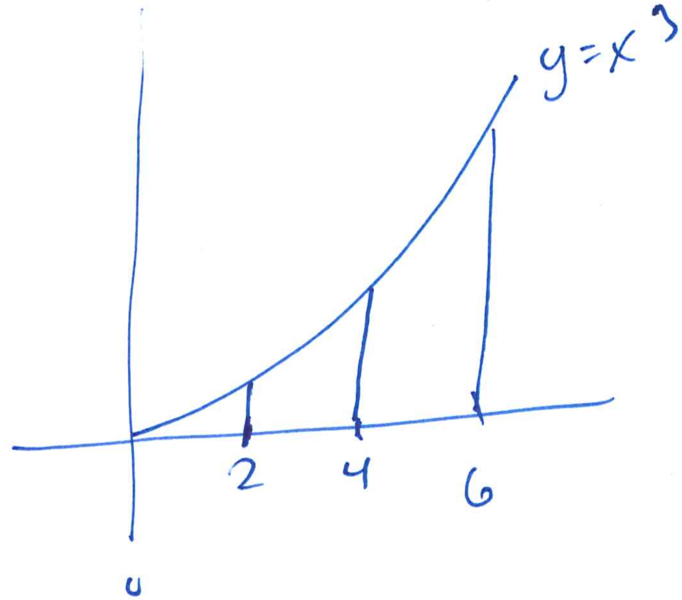
is called a Riemann Sum of f on $[a,b]$.

It's called a: $\left(\begin{array}{l}\text{left}\\ \text{right}\\ \text{midpoint}\end{array}\right)$ Riemann sum if

x_i^* is the (") of the interval $[x_{i-1}, x_i]$.

ex $y=x^3$

Approx area under curve, on $[0,6]$, using 3 subintervals + Riemann Sum.



$n=3$

$\Delta x = \frac{6-0}{3} = 2$

width of each partition

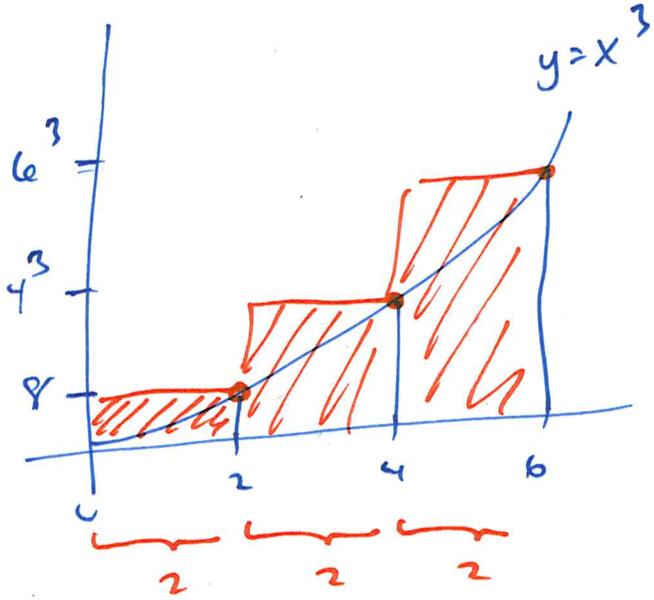
Grid points: 0, 2, 4, 6

Riemann Sum:

$\Delta x\, f(x_1^*) + \Delta x\, f(x_2^*) + \Delta x\, f(x_3^*)$

$= 2 f(x_1^*) + 2 f(x_2^*) + 2 f(x_3^*)$

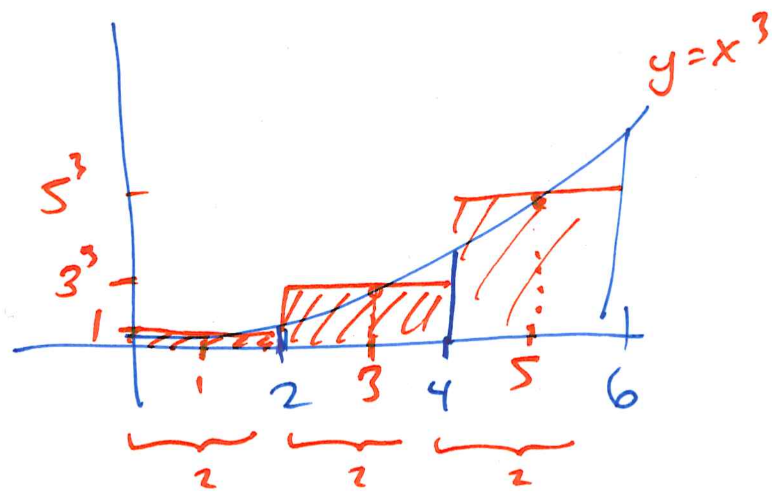
Right RS:



$$\boxed{2(8) + 2(4^3) + 2(6^3)}$$

Approximation of area under curve (Right RS)

Midpoint RS:



$$\boxed{(2)(1) + (2)(3^3) + (2)(5^3)}$$

Approx of area under curve: midpoint RS

(ex) Suppose a car's speed is:

time	12:00	12:15	12:30	12:45	1:00
speed	60 kph	80 kph	100 kph	100 kph	40 kph

How far did the car travel from 12:00 to 1:00?

$\Delta x = \frac{1}{4}$
$n=4$

Left RS:

$$\left(\tfrac{1}{4}\right)(60) + \left(\tfrac{1}{4}\right)(80) + \left(\tfrac{1}{4}\right)(100) + \tfrac{1}{4}(100)$$

Using Riemann Sum:

How many intervals? (n)

What is Δx?

Write out sum

Left, Right, MP

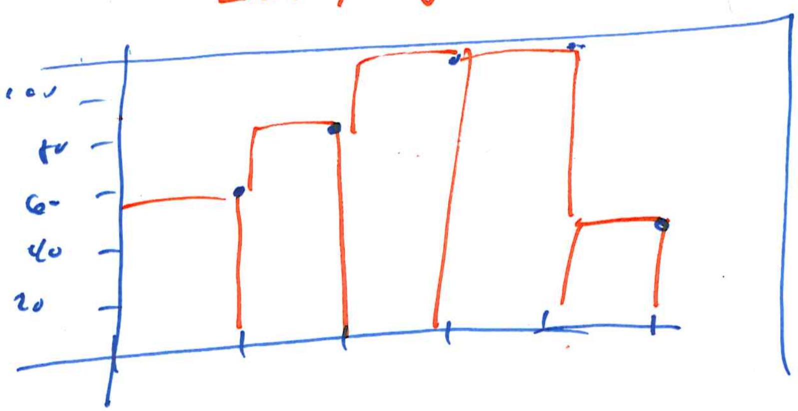
$\Delta x = \frac{1}{4}$
$n=4$

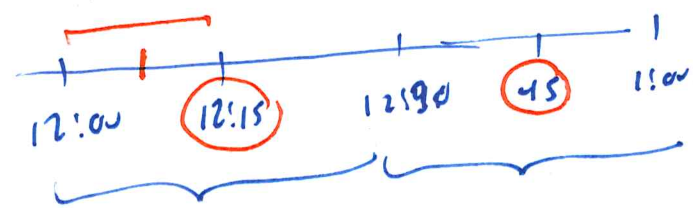
Right RS:

$$\left(\tfrac{1}{4}\right)(80) + \tfrac{1}{4}(100) + \tfrac{1}{4}(100) + \tfrac{1}{4}(40)$$

$n=2$
$\Delta x = \frac{1}{2}$

Midpoint RS:





$$\tfrac{1}{2}(80) + \tfrac{1}{2}(100)$$

Quick Note:

The gradient of a function $f(x,y)$ is the vector

$$\nabla f = \langle f_x, f_y \rangle$$

ex $f(x,y) = x^2 + y^2$

$$\nabla f = \langle 2x, 2y \rangle$$

$$\nabla f = \lambda \nabla g$$

$$\Downarrow$$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases}$$

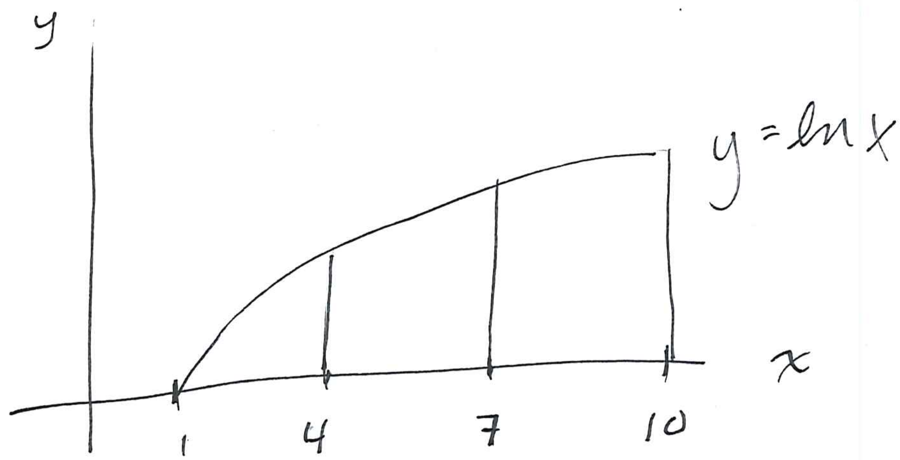
Riemann Sums

(ex) Want to approximate area under $y = \ln x$

from $x = 1$ to $x = 10$

using Riemann Sums, 3 subintervals

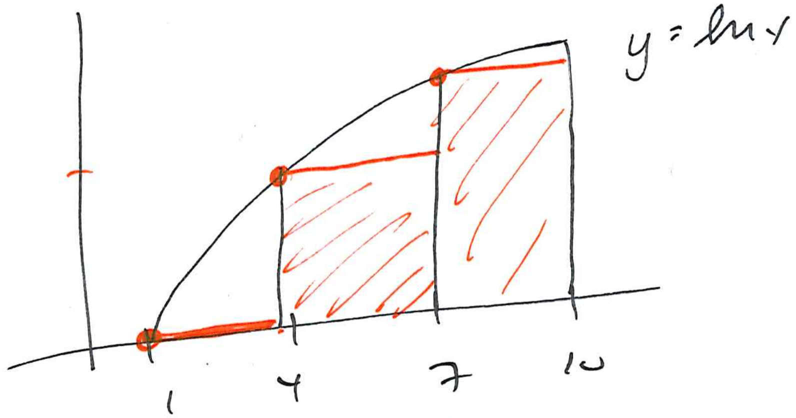
(3 rectangles)



Δx : width of each □

$\Delta x = \frac{10-1}{3} = 3$

Left:



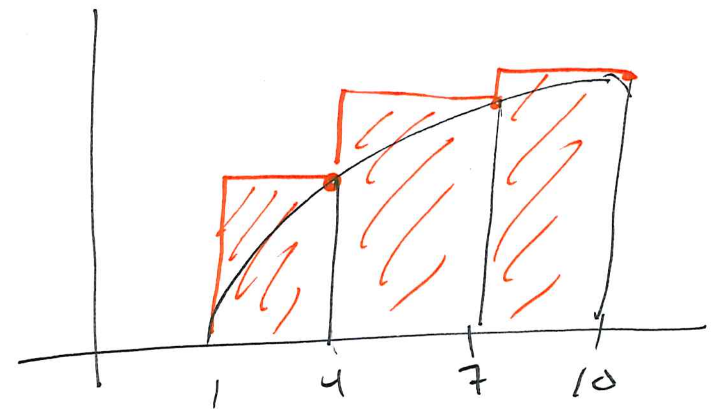
1st □: width 3
height 0

2nd □: width 3
height ln 4

3rd □: width 3
height ln 7

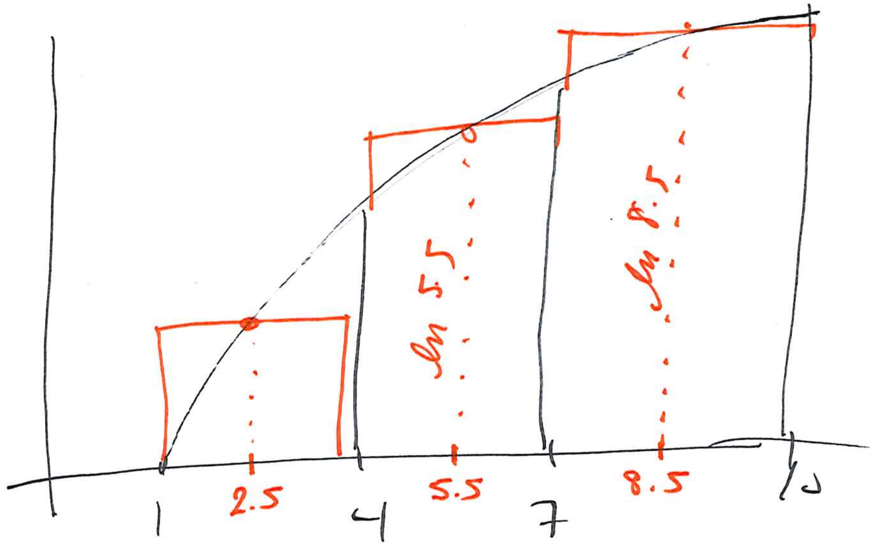
RS: $(3)(0) + (3)\ln 4 + (3)\ln 7$

Right:



RS: $(3)\ln 4 + (3)\ln 7 + 3\ln 10$

Midpoint



RS: $(3)\ln 2.5 + 3\ln 5.5 + 3\ln 8.5$

Review of Σ-notation

$$\sum_{k=a}^{b} f(k)$$

ex: $\displaystyle\sum_{k=-2}^{1} (2k+5) = \boxed{\underset{(k=-2)}{(-4+5)} + \underset{(k=-1)}{(-2+5)} + \underset{(k=0)}{(5)} + \underset{(k=1)}{(2+5)}}$

ex $\displaystyle\sum_{k=6}^{8} (k^2-k) = \boxed{(36-6)+(49-7)+(64-8)}$

Which are OK?

✓ (A) $\sum_{k=1}^{15} (k^2 - k) = \sum_{k=1}^{15} k^2 - \sum_{k=1}^{15} k$

Order doesn't matter in addition

$1^2 - 1 + 2^2 - 2 + 3^2 - 3 \cdots$

$= (1^2 + 2^2 + 3^2 + \cdots) - (1 + 2 + 3 \cdots)$

✓ (B) $\sum_{k=1}^{15} 3k = 3 \sum_{k=1}^{15} k$ Factoring

$3(1) + 3(2) + 3(3) + \cdots + 3(15)$

$= 3[1 + 2 + 3 + \cdots + 15]$

(C) $\sum_{k=1}^{15} k(k-1) \neq k \sum_{k=1}^{15} (k-1)$

$1(0) + 2(1) + 3(2) + \cdots$

✓ (D) $\sum_{k=1}^{15} (k-1) = -15 + \sum_{k=1}^{15} k$

$\parallel$

$(1-1) + (2-1) + (3-1) + (4-1) + \cdots + (15-1)$

$\parallel$

$(1+2+3+4+\cdots+15) \underbrace{-1 \ -1 \ -1 \ -1 \ \cdots \ -1}_{15 \text{ times}} = \sum_{k=1}^{15} k - \sum_{k=1}^{15} 1$

$\parallel$

$\sum_{k=1}^{15} k - 15$

ex) Write in Σ-notation:

$$2+3+4+5+6+7 = \sum_{k=2}^{7} k = \sum_{k=1}^{6} (k+1)$$

$$4+6+8+10+12 = \sum_{k=2}^{6} 2k \quad = \quad \sum_{k=2}^{6} k+\cancel{2} = 4+\cancel{8}$$

$k=2 \quad k=3$

$$5+7+9+11+13 = \sum_{k=2}^{6} (2k+1)$$

$$3.5+6.5+9.5+12.5+15.5 = \sum_{k=1}^{5} \left(3k+\tfrac{1}{2}\right)$$

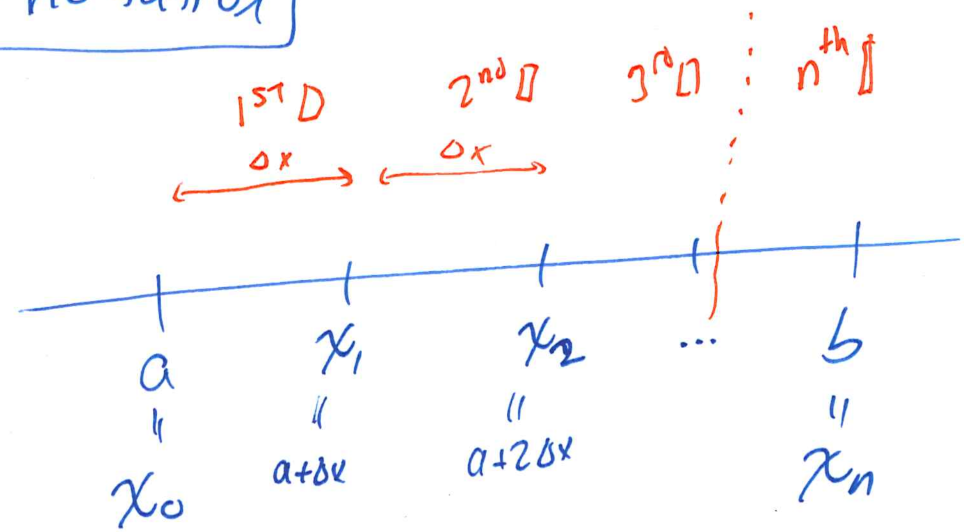
$$\underset{2^{-1}}{\tfrac{1}{2}} + \underset{2^{0}}{1} + \underset{2^{1}}{2} + \underset{2^{2}}{4} + \underset{2^{3}}{8} + \underset{2^{4}}{16} + \underset{2^{5}}{32} = \sum_{k=-1}^{5} 2^k$$

$$-\tfrac{1}{2} + \underset{(-2)^0}{1} - \underset{(-2)^1}{2} + \underset{(-2)^2}{4} - \underset{(-2)^3}{8} + 16 - 32 = \sum_{k=-1}^{5} (-2)^k$$

$$\sum_{k=2}^{6} (2k+1) = 5+7+9+11+13$$

Riemann Sums in Σ-notation

function $f(x)$
over $[a,b]$
use n subintervals
(n rectangles)

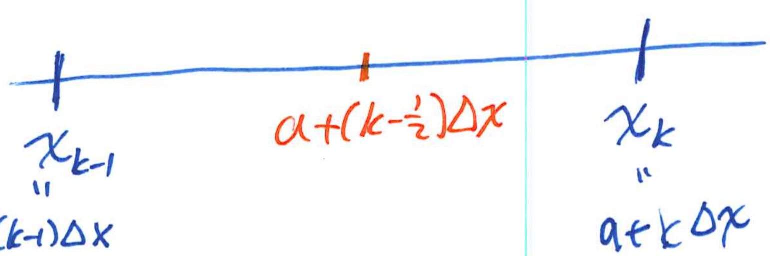


Bases of all rectangles:

$$\Delta x = \frac{b-a}{n}$$

Height: 1) depends on left/right/middle Riemann sum
2) depends on which ▯

(k^{th} ▯)



Area of k^{th} $\square$:

$(\text{base})(\text{height})$

$\Delta x\,(\text{height})$

Left: $\Delta x\, f(x_{k-1}) = \Delta x\, f(a+(k-1)\Delta x)$

Right: $\Delta x\, f(x_k) = \Delta x \cdot f(a+k\Delta x)$

Midpt: $\Delta x \cdot f\left(a+(k-\tfrac{1}{2})\Delta x\right)$

where: $\Delta x = \dfrac{b-a}{n}$

General Formulas for Riemann Sums:

Left: $\sum_{k=1}^{n} \Delta x \cdot f(a+(k-1)\Delta x)$ $\qquad \Delta x = \frac{b-a}{n}$

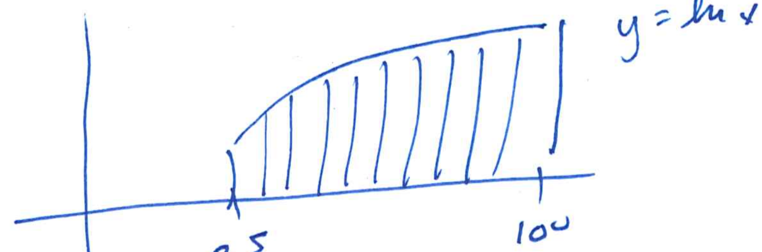
Right: $\sum_{k=1}^{n} \Delta x \cdot f(a+k\Delta x)$ $\qquad \Delta x = \frac{b-a}{n}$

Midpoint: $\sum_{k=1}^{n} \Delta x \cdot f(a+(k-\frac{1}{2})\Delta x)$

(ex) $f(x) = \ln x$

$[25, 100]$

$n = 10$ rectangles



$\Delta x = \frac{100-25}{10} = \frac{75}{10} = 7.5$

$a = 25$

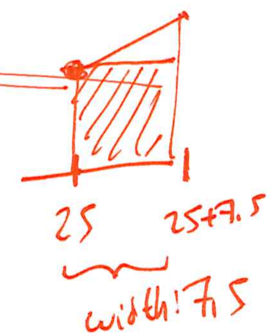
Left RS:

$$\sum_{k=1}^{10} \Delta x \cdot f(a+(k-1)\Delta x)$$

$$= \sum_{k=1}^{10} (7.5) \ln(25+(k-1)\cdot 7.5)$$ some number

Right RS: $\sum_{k=1}^{10} (7.5) \cdot \ln(25+k \cdot 7.5)$ some number

Midpt RS: $\sum_{k=1}^{10} (7.5) \ln(25+(k-\frac{1}{2})7.5)$ some number

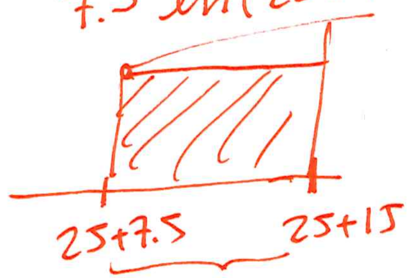


Left RS first terms: $(k=1)$ $7.5 \ln(25+0) = 7.5 \ln 25$

$(k=2)$ $7.5 \ln(25+7.5)$

area of 2^{nd} ▯

etc....



Low-Degree Powers of k work nicely with Σ

$\boxed{\text{p. 340}}$ text

- $\displaystyle\sum_{k=1}^{n} c = \underbrace{c + c + c + \cdots + c}_{n \text{ times}} = cn$

  constant
  $k^0 = 1$

- $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$

  $= \dfrac{n(n+1)}{2}$

- $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

- $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

$1 + 2 + 3 + \cdots + 49 + 50 + 51 + 52 + \cdots + 97 + 98 + 99 + 100$

$100 = n$

101 101 101 101 101

Sum: $101 \cdot 50 = (n+1)\left(\frac{n}{2}\right)$

(ex) Evaluate Rieman Sum (right)

$f(x) = x^2 + x$

$[1, 6]$

$n = 100$ rectangles

$\Delta x = \frac{6-1}{100} = \frac{5}{100} = \frac{1}{20}$

(width)

heigt: $f(x_k) = f(a + k\Delta x) = f(1 + k\cdot\frac{1}{20})$

right endpt

Sum: $\sum_{k=1}^{100} \underbrace{\frac{1}{20}}_{\substack{\Delta x \\ \text{width}}} \cdot \underbrace{f(\overbrace{1+\frac{k}{20}}^{x})}_{\substack{\text{height} \\ @ \\ \text{right} \\ \text{endpt}}} = \sum_{k=1}^{100} \frac{1}{20}\cdot\left[\left(1+\frac{k}{20}\right)^2 + \left(1+\frac{k}{20}\right)\right]$

$= \frac{1}{20}\sum_{k=1}^{100}\left(1 + \frac{2k}{20} + \frac{k^2}{20^2} + 1 + \frac{k}{20}\right) = \frac{1}{20}\sum_{k=1}^{100}\left(2 + \frac{3}{20}k + \frac{1}{20^2}k^2\right)$

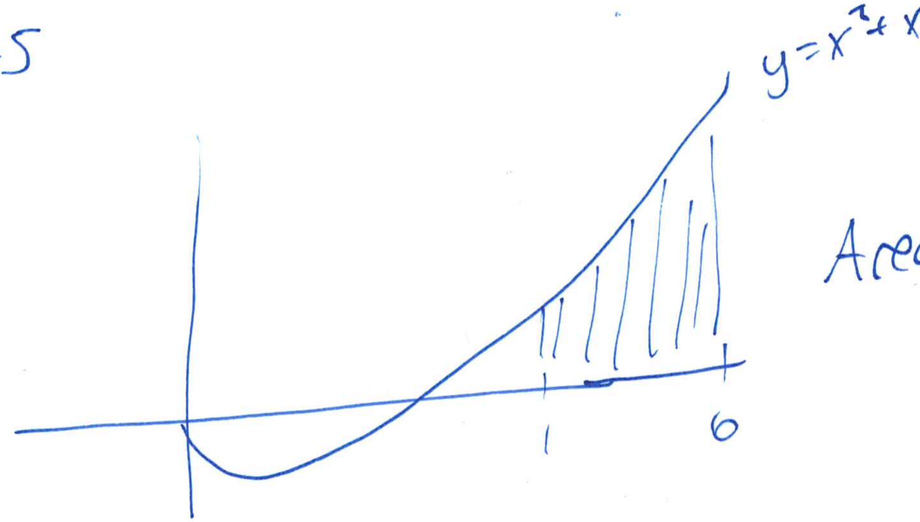
$$= \frac{1}{20}\left[\sum_{k=1}^{100} 2 + \sum_{k=1}^{100} \frac{3}{20}k + \sum_{k=1}^{100} \frac{1}{20^2}k^2\right]$$

$$= \frac{1}{20}\left[200 + \frac{3}{20}\sum_{k=1}^{100} k + \frac{1}{20^2}\sum_{k=1}^{100} k^2\right]$$

use list!

$$= \frac{1}{20}\left[200 + \frac{3}{20}\left(\frac{101\cdot 100}{2}\right) + \frac{1}{20^2}\left(\frac{100\cdot 101\cdot 201}{6}\right)\right.$$

$$= 90.16875$$



Area $\approx$ 90.16875

Riemann Sums

ex Approx area under $f(x) = (x+2)^2$ on $[3,5]$ using $n=100$ rectangles.

Use right Riemann Sum

General formula for right RS: (n □, interval $[a,b]$)

$$\sum_{k=1}^{n} \Delta x \cdot f(a+k\Delta x) \quad \text{where } \Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{5-3}{100} = \frac{2}{100} = \frac{1}{50}$$

$a=3 \quad n=100$

Riemann Sum: $$\sum_{k=1}^{100} \frac{1}{50} f\Big(\underbrace{3+k\left(\tfrac{1}{50}\right)}_{x}\Big) = \sum_{k=1}^{100} \frac{1}{50}\left(2+3+\frac{1}{50}k\right)^2$$

$f(x) = (x+2)^2$

$$= \sum_{k=1}^{100} \frac{1}{50}\left(5+\frac{1}{50}k\right)^2 = \sum_{k=1}^{100} \frac{1}{50}\left(25+\underbrace{2(5)\cdot\frac{1}{50}k}_{\frac{1}{5}k}+\frac{1}{50^2}k^2\right)$$

$$= \sum_{k=1}^{100} \left(\frac{1}{2} + \frac{1}{50 \cdot 5} k + \frac{1}{50^3} k^2 \right)$$

$$= \underbrace{\sum_{k=1}^{100} \frac{1}{2}} + \sum_{k=1}^{100} \frac{1}{50 \cdot 5} k + \sum_{k=1}^{100} \frac{1}{50^3} k^2$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

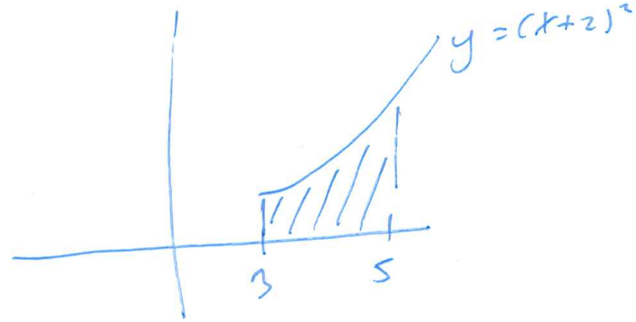
$$= 100\left(\tfrac{1}{2}\right) + \frac{1}{50 \cdot 5} \underbrace{\sum_{k=1}^{100} k}_{\text{formula}} + \frac{1}{50^3} \underbrace{\sum_{k=1}^{100} k^2}_{\text{formula}}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= 50 + \frac{1}{50 \cdot 5} \cdot \left(\frac{100 \cdot 101}{2} \right) + \frac{1}{50^3} \cdot \left(\frac{100 \cdot 101 \cdot 201}{6} \right)$$

$$\underset{\text{(calculator)}}{=} \boxed{72.9068}$$

Approx (right RS) area:



ex Area under $f(x) = (x+2)^2$, over $[3, 5]$
using $n=100$ rectangles:
Midpoint RS

General formula for midpt RS:

$$\sum_{k=1}^{n \; (100)} \Delta x \cdot f\left(a + \left(k - \tfrac{1}{2}\right)\Delta x\right)$$

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{5-3}{100} = \frac{2}{100} = \frac{1}{50}$$

$a = 3$

$$\sum_{k=1}^{100} \frac{1}{50} \cdot f\left(3 + \left(k - \tfrac{1}{2}\right)\tfrac{1}{50}\right)$$

$$= \sum_{k=1}^{100} \frac{1}{50} f\left(3 - \frac{1}{100} + \frac{1}{50}k\right)$$

$$= \sum_{k=1}^{100} \frac{1}{50} \cdot f\left(\frac{299}{100} + \frac{1}{50}k\right)$$

$$= \sum_{k=1}^{100} \frac{1}{50} \cdot \left(2 + \frac{299}{100} + \frac{1}{50}k\right)^2$$

$$= \sum_{k=1}^{100} \frac{1}{50}\left(\frac{499}{100} + \frac{1}{50}k\right)^2$$

$$= \sum_{k=1}^{100} \frac{1}{50}\left(\left(\frac{499}{100}\right)^2 + 2\left(\frac{499}{100}\right)\cdot\frac{1}{50}k + \frac{1}{50^2}k^2\right)$$

$$= \sum_{k=1}^{100}\left(\frac{499^2}{100^2\cdot 50} + \frac{2\cdot 499}{100\cdot 50^2}k + \frac{1}{50^3}k^2\right)$$

$$= \underbrace{\sum_{k=1}^{100} \frac{499^2}{100^2\cdot 50}} + \sum_{k=1}^{100} \frac{2\cdot 499}{100\cdot 50^2}k + \sum_{k=1}^{100} \frac{1}{50^3}k^2$$

$$= \cancel{100}\left(\frac{499^2}{100^{\cancel{2}}\cdot 50}\right) + \frac{\cancel{2}\cdot 499}{\cancel{100}_{50}\cdot 50^2}\sum_{k=1}^{100} k + \frac{1}{50^3}\sum_{k=1}^{100} k^2$$

$$= \frac{499^2}{100\cdot 50} + \frac{499}{50^3}\sum_{k=1}^{100} k + \frac{1}{50^3}\sum_{k=1}^{100} k^2$$

$$= \frac{499^2}{100\cdot 50} + \frac{499}{50^3}\left(\frac{100\cdot 101}{2}\right) + \frac{1}{50^3}\cdot\left(\frac{100\cdot 101\cdot 201}{6}\right)$$

Formulas (p 340)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad (n=100)$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

CALCULATOR: 72.6666

ex Find exact area under $y=(x+2)^2$; $[3,5]$

Plan: • take RS using n rectangles ← might as well use easiest RS: right RS

• evaluate (in terms of n)

• take limit as $n \to \infty$

$\Delta x = \frac{b-a}{n} = \frac{2}{n}$

General form: $\sum_{k=1}^{n} \Delta x \, f(a + k\Delta x)$

$a = 3$

$$= \sum_{k=1}^{n} \left(\frac{2}{n}\right) \cdot f\left(3 + k \cdot \frac{2}{n}\right) = \sum_{k=1}^{n} \left(\frac{2}{n}\right)\left(2 + 3 + \frac{2}{n}k\right)^2$$

$$= \sum_{k=1}^{n} \left(\frac{2}{n}\right)\left(5 + \frac{2}{n}k\right)^2 = \sum_{k=1}^{n} \left(\frac{2}{n}\right)\left(25 + 2(5)\left(\frac{2}{n}k\right) + \left(\frac{2}{n}\right)^2 k^2\right)$$

$$= \sum_{k=1}^{n} \left(\frac{50}{n} + \frac{40}{n^2}k + \left(\frac{2}{n}\right)^3 k^2\right)$$

$$= \sum_{k=1}^{n} \frac{50}{n} + \sum_{k=1}^{n} \frac{40}{n^2} k + \sum_{k=1}^{n} \frac{8}{n^3} k^2$$

$$= \sum_{k=1}^{n} \frac{50}{n} + \frac{40}{n^2} \sum_{k=1}^{n} k + \frac{8}{n^3} \sum_{k=1}^{n} k^2$$

Formulas: (p 340)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= (\cancel{n})\left(\frac{50}{\cancel{n}}\right) + \frac{\overset{20}{\cancel{40}}}{\cancel{n^2}} \cdot \frac{\cancel{n}(n+1)}{\cancel{2}} + \frac{\overset{4}{\cancel{8}}}{n^{\cancel{3}\,2}} \cdot \frac{\cancel{n}(n+1)(2n+1)}{\underset{3}{\cancel{6}}}$$

$$= 50 + 20 \cdot \left(\frac{n+1}{n}\right) + \frac{4}{3}\left(\frac{(n+1)(2n+1)}{n^2}\right)$$

$$= 50 + 20\left(1 + \frac{1}{n}\right) + \frac{4}{3}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= 50 + 20\left(1 + \frac{1}{n}\right) + \frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

$$\xrightarrow[n \to \infty]{} 50 + 20(1+0) + \frac{4}{3}(1+0)(2+0)$$

$$= 50 + 20 + \frac{8}{3}$$

$$= 70 + 8/3$$

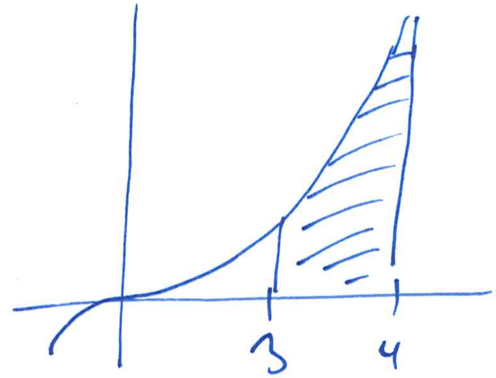
$$= 70 + 2 + 2/3 = \boxed{72 + 2/3} = 72.\overline{66}$$

exact area under $y = (x+2)^2$ from $x = 3$ to $x = 5$

(ex) Find the exact area under the curve

$y = x^3$

over the interval $[3,4]$



General Right Riemann Sum:

$$\sum_{k=1}^{n} \Delta x \, f(a + k\Delta x)$$

$$\Delta x = \frac{b-a}{n} = \frac{4-3}{n} = \frac{1}{n}$$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

formulas:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=1}^{n} \frac{1}{n} f\left(3 + k\left(\frac{1}{n}\right)\right)$$

$$= \sum_{k=1}^{n} \frac{1}{n} \cdot \left(3 + \frac{k}{n}\right)^3$$

$$= \sum_{k=1}^{n} \frac{1}{n} \left[27 + 3 \cdot 9 \cdot \frac{k}{n} + 3 \cdot 3 \cdot \left(\frac{k}{n}\right)^2 + \left(\frac{k}{n}\right)^3\right]$$

$$= \sum_{k=1}^{n} \frac{1}{n}\left[27 + \frac{27}{n}k + \frac{9}{n^2}k^2 + \frac{1}{n^3}k^3\right]$$

$$= \sum_{k=1}^{n}\left(\frac{27}{n} + \frac{27}{n^2}k + \frac{9}{n^3}k^2 + \frac{1}{n^4}k^3\right)$$

$$= \sum_{k=1}^{n}\frac{27}{n} + \sum_{k=1}^{n}\frac{27}{n^2}k + \sum_{k=1}^{n}\frac{9}{n^3}k^2 + \sum_{k=1}^{n}\frac{1}{n^4}k^3$$

FORMULAS

$$= n\left(\frac{27}{n}\right) + \frac{27}{n^2}\sum_{k=1}^{n}k + \frac{9}{n^3}\sum_{k=1}^{n}k^2 + \frac{1}{n^4}\sum_{k=1}^{n}k^3$$

$$= 27 + \frac{27}{n^{\cancel{2}\,1}}\left(\frac{\cancel{n}(n+1)}{2}\right) + \frac{\cancel{9}\,3}{n^{\cancel{3}\,2}}\left(\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}\,2}\right) + \frac{1}{n^{\cancel{4}\,2}}\left(\frac{\cancel{n^2}(n+1)^2}{4}\right)$$

$$= 27 + \frac{27}{2}\left(\frac{n+1}{n}\right) + \frac{3}{2}\left(\frac{(n+1)}{n}\cdot\frac{(2n+1)}{n}\right) + \frac{1}{4}\left(\frac{n+1}{n}\right)^2$$

$$= 27 + \frac{27}{2}\left(1+\frac{1}{n}\right) + \frac{3}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{1}{4}\left(1+\frac{1}{n}\right)^2$$

$$\xrightarrow[n\to\infty]{} 27 + \frac{27}{2}(1+0) + \frac{3}{2}(1+0)(2+0) + \frac{1}{4}(1+0)^2$$

$$= 27 + \frac{27}{2} + 3 + \frac{1}{4} = 30 + \frac{27}{2} + \frac{1}{4} = 30 + 13 + \frac{1}{2} + \frac{1}{4} = 43 + \frac{3}{4}$$

$$= \boxed{43.75}$$

To find exact area under $y = f(x)$, $[a,b]$:

$$\text{Area} = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \Delta x\, f(x_k^*)\right)$$

could be left/right/MP

$$= \lim_{n\to\infty}\left(\sum_{k=1}^{n} \Delta x\, f(a+k\Delta x)\right)$$

Last time, we said:

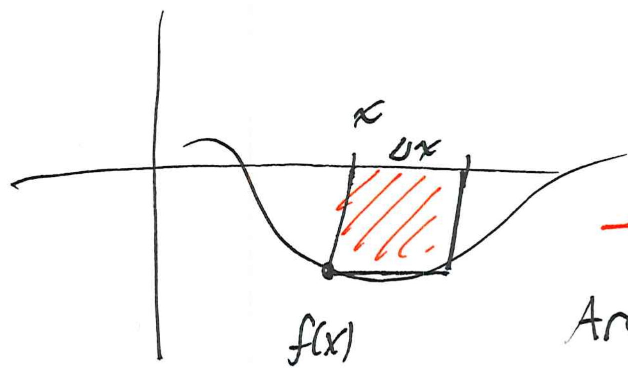
Area under curve $y = f(x)$, $[a, b]$ is

$$\lim_{n \to \infty} \sum_{k=1}^{n} \Delta x \cdot f(x_k^*), \quad \Delta x = \frac{b-a}{n}$$

x_k^* is between

$a+(k-1)\Delta x$ + $a+k\Delta x$

Q: What if $f(x) < 0$?



$-$Area

Area: (width)(height)

$= \Delta x \, |f(x)|$

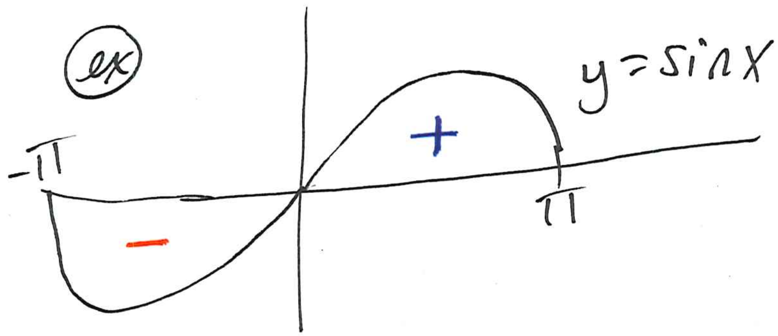
$= \Delta x \, (-f(x))$

$= -\Delta x \, f(x)$

Actually calculating: "net area"

Area above axis

$-$ area below axis



$f(x) = \sin x$

$[-\pi, \pi]$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Delta x \cdot f(x_k^*) = 0$$

(positive area exactly cancels out negative area)

Notation: Definite Integrals

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x \cdot f(x_k^*), \quad \text{where } \Delta x = \frac{b-a}{n} \text{ etc.}$$

dx : "differential"
$\lim_{n\to\infty} \Delta x$

a, b : bounds
w/o bounds, integral is "indefinite"

$\int$ "integral sign"
elongated s for "sum"
same as Σ

Properties of Definit Integral

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

   why? $\Delta x = \frac{b-a}{n} = -\left[\frac{a-b}{n}\right]$

2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)dx + \int_a^b g(x)\,dx$

   why? $\sum_{k=1}^{n}(f+g) = \sum_{k=1}^{n} f + \sum_{k=1}^{n} g$

3. $\int_a^b c\cdot f(x)dx = c\int_a^b f(x)dx$

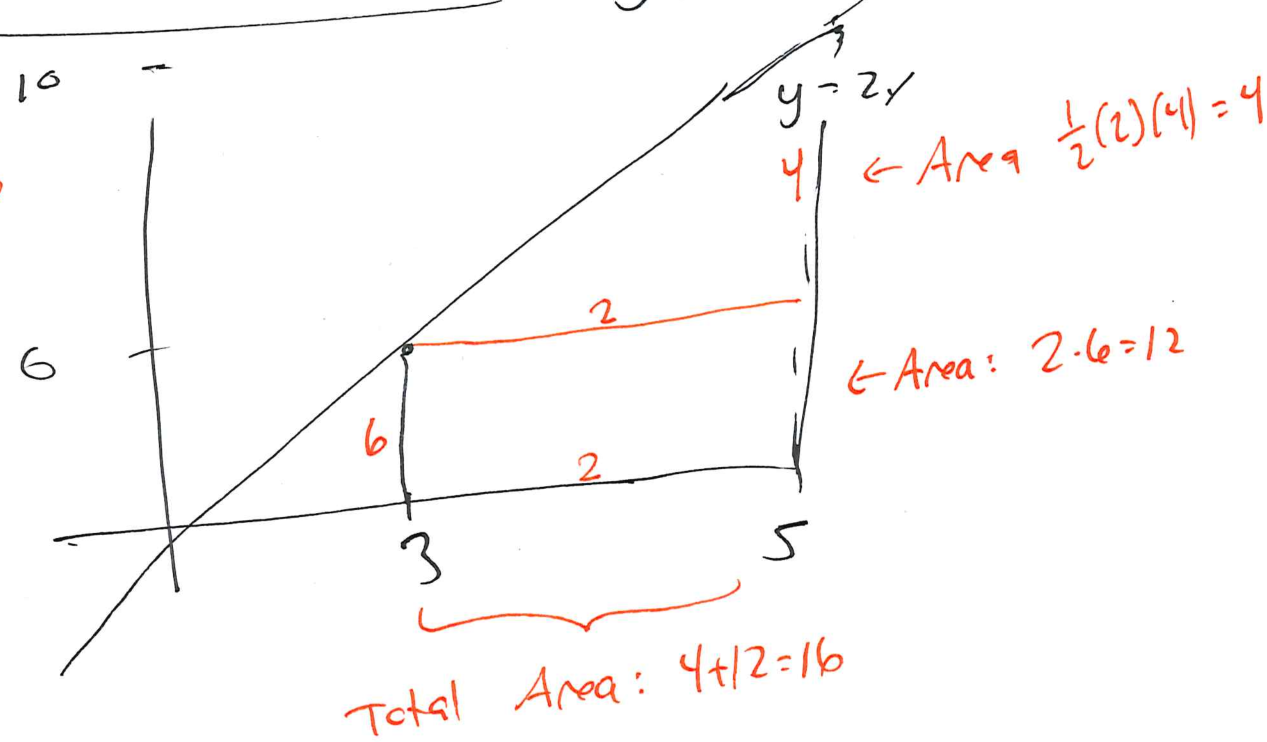
   c-constant

4. $\int_{\underline{a}}^{\underline{b}} f(x)dx = \int_{\underline{a}}^{m} f(x)dx + \int_m^{\underline{b}} f(x)dx$

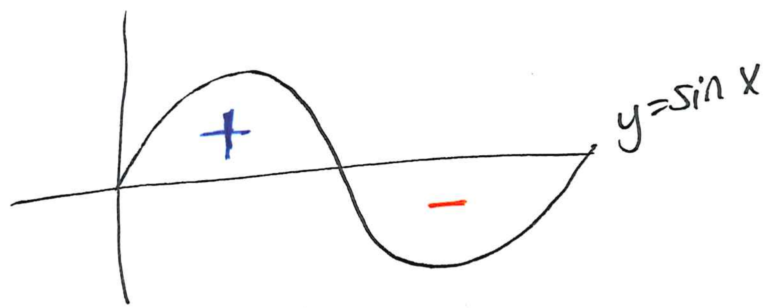
why?

Evaluating Definite Integrals Using Geometry

ex) $\int_3^5 2x\,dx = 16$



ex) $\int_0^{2\pi} \sin x\,dx = 0$

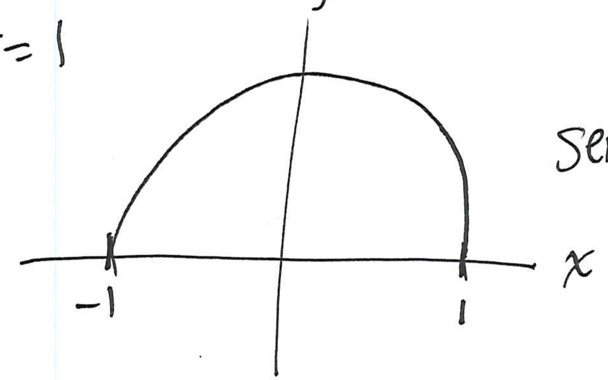


ex $\int_{-1}^{1} \sqrt{1-x^2}\,dx = \pi/2$

$y = \sqrt{1-x^2}$ $\quad y \geq 0$

$y^2 = 1-x^2$

$x^2+y^2 = 1$

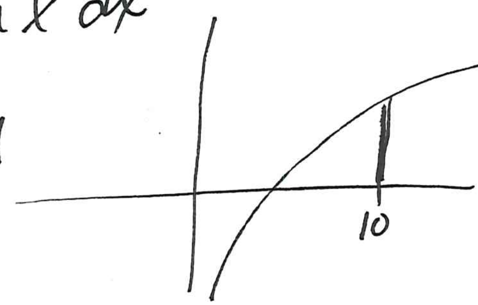


semicircle

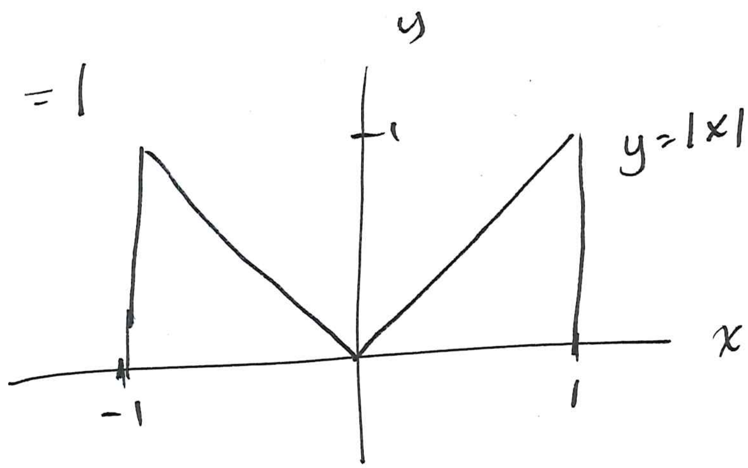
$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(1) = \pi/2$

ex $\int_{10}^{10} \ln x\,dx = 0$

no width!



ex $\int_{-1}^{1} |x|\,dx = 1$

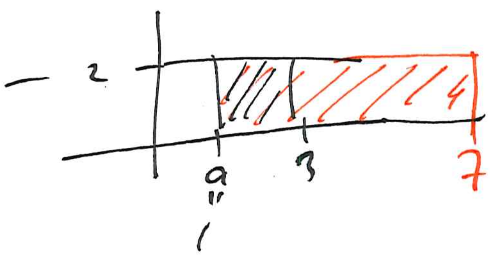


Ch 5.3 Fundamental Theorem of Calculus

Area function: $A(x) = \int_a^x f(t)\,dt$

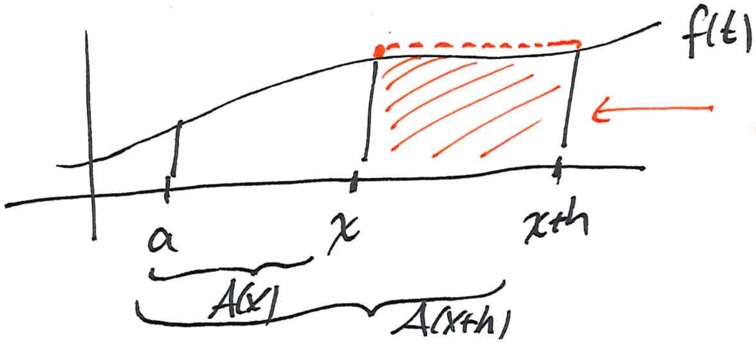
(ex) If $f(t) = 2$ $a = 1$

$A(3) = 4$ $A(7) = 12$



Derivative of Area Function:

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} \frac{h \cdot f(x)}{h} = f(x)$$



$A(x+h) - A(x) \approx h \cdot f(x)$

As $h \to 0$, $A(x+h) - A(x) \longrightarrow h \cdot f(x)$

Fundamental Theorem of Calculus, Part 1:

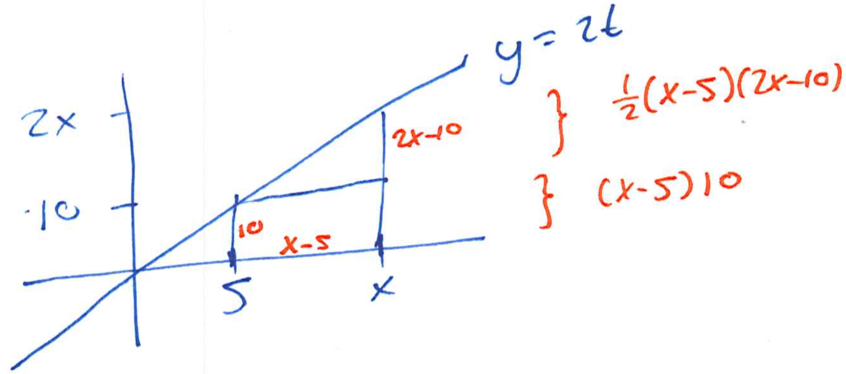
If f is continuous on $[a,b]$, then the area function

$A(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$

and differentiable on (a,b), and

$$A'(x) = f(x)$$

ex $A(x) = \int_5^x 2t\,dt$



$= 10(x-5) + \frac{1}{2}(x-5)(2x-10)$

$= 10(x-5) + (x-5)(x-5)$

$= (x-5)(10+x-5)$

$= (x-5)(x+5)$

$= x^2 - 25$

NOTICE: $A(x) = x^2 - 25$

$A'(x) = 2x = f(x)$

We are considering $A(x) = \int_p^x f(t)\,dt$

FTC(I): $A'(x) = f(x)$

Lots of functions have $f(x)$ as derivative.

Take any function $F(x)$ such that $F'(x) = f(x)$.

Since A, F have same derivative, they only differ by some constant, say C:

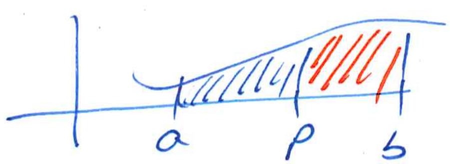
$$F(x) = A(x) + C$$

$$F(b) - F(a) = [A(b) + \cancel{C}] - [A(a) + \cancel{C}] = A(b) - A(a)$$

Notice:

$$= \int_p^b f(t)\,dt - \int_p^a f(t)\,dt = \int_p^b f(t)\,dt + \int_a^p f(t)\,dt$$

$$= \int_a^p f(t)\,dt + \int_p^b f(t)\,dt = \int_a^b f(t)\,dt$$



Fundamental Theorem of Calculus, Part II

If f is continuous on $[a,b]$, and F is <u>any</u> antiderivative of f (I mean: $F'(x) = f(x)$), then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

ex $\int_5^{10} 2x\,dx = 10^2 - 5^2 = \boxed{75}$

could use geometry
could also use FTC

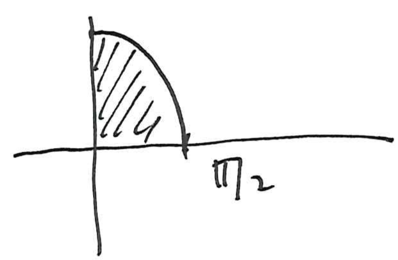
$f(x) = 2x$

$F(x) = x^2$

ex $\int_0^{\pi/2} \cos x\,dx = \sin(\pi/2) - \sin(0)$
$= 1 - 0 = \boxed{1}$

$f(x) = \cos x$

$F(x) = \sin x$



$$\int \cos x \, dx = \sin x + C$$

$$\int 15 \cos x \, dx = 15 \sin x + C$$

Constant Powers of x:

$f(x)$	$f'(x)$
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^n	nx^{n-1}

$f(x)$	$\int f(x)\,dx$
1	$x + C$
x	$\frac{1}{2}x^2 + C$
x^2	$\frac{1}{3}x^3 + C$
x^3	$\frac{1}{4}x^4 + C$
x^n	$\frac{1}{n+1}x^{n+1} + C$ if $n \neq -1$
$x^{-1} = \frac{1}{x}$	$\ln\lvert x\rvert + C$

$$\int \sqrt{x}\, dx = \int x^{1/2} dx = \frac{2}{3} x^{3/2} + C$$

indefinite (no bounds)

$$\int_1^9 \sqrt{x}\, dx = \frac{2}{3}(9)^{3/2} - \frac{2}{3}(1)^{3/2} = \frac{2}{3} \cdot \sqrt{9}^3 - \frac{2}{3} = \frac{2}{3} \cdot 27 - 2/3$$

$$= \frac{2}{3}(26)$$

definite integral - area

ex $\quad \frac{d}{dx}\{\arctan x\} = \frac{1}{1+x^2}$, so: $\int \frac{1}{1+x^2}\,dx = \arctan x + C$

Note: This is a little surprising:

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = -x^{-1} + C = \frac{-1}{x} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\frac{d}{dx}\left\{\frac{1}{a}\arctan\left(\frac{x}{a}\right)\right\} = \frac{1}{a}\cdot\frac{1}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1}{a} = \frac{1}{a^2\left(1+\frac{x^2}{a^2}\right)} = \frac{1}{a^2+x^2}$$

(a constant)

So: $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

ex) $\frac{d}{dx}\{\arcsin x\} = \frac{1}{\sqrt{1-x^2}}$

$$\frac{d}{dx}\left\{\arcsin\left(\frac{x}{a}\right)\right\} = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{1}{a} = \frac{1}{\sqrt{a^2}\cdot\sqrt{1-\frac{x^2}{a^2}}}$$

$(a>0)$

$$= \frac{1}{\sqrt{a^2 - \cancel{a^2}\cdot\frac{x^2}{\cancel{a^2}}}} = \frac{1}{\sqrt{a^2-x^2}}$$

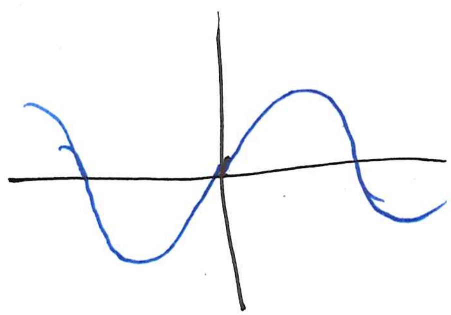
So: $\boxed{\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C}$

Even + Odd Functions

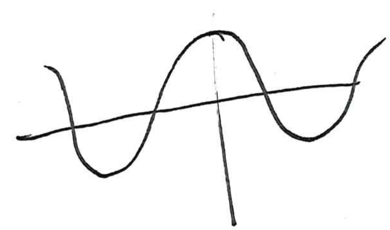
Odd function: $f(-x) = -f(x)$

ex: $f(x) = \sin x$

$f(x) = x^3 \longrightarrow f(-2) = -8 = -f(2)$



Odd Function: $\int_{-a}^{a} f(x)dx = 0$



Even function: $f(-x) = f(x)$

ex: $\cos x$

$f(x) = x^2 \qquad f(-2) = 4 = f(2)$

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

Substitution Rule

(Chain Rule in reverse)

ex $f(x) = \sin(3x^2+x)$

$f'(x) = \cos(3x^2+x)\cdot(6x+1)$

$$\int \cos(\underbrace{3x^2+x}_{\text{inside}})\cdot(\underbrace{6x+1}_{\text{deriv. of inside}})\,dx = \sin(3x^2+x)+C$$

ex Chain rule:

$$\frac{d}{dx}\{f(g(x))\} = f'(g(x))\cdot g'(x)$$

Backwards:

$$\int f'(g(x))\cdot g'(x)\,dx = f(g(x))+C$$

Mnemonic: change of variable

"dictionary"

$\boxed{g(x) = u}$

$\frac{du}{dx} = g'(x)$

$\boxed{du = g'(x)\,dx}$

$$\int f'(g(x))\cdot \underbrace{g'(x)\,dx}$$ =

$$\int f'(u)\cdot du = f(u)+C$$

$$= f(g(x))+C$$

ex

$$\int e^{\boxed{\sin x}} \underbrace{\cos x\, dx}_{du} = \int e^u\, du = e^u + c = \boxed{e^{\sin x} + c}$$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\, dx$

Check: $\frac{d}{dx}\{e^{\sin x} + c\} = e^{\sin x} \cdot \cos x$ ✓

ex

$$\int e^x \sin(e^x)\, dx = \int \sin u\, du = -\cos u + c = \boxed{-\cos(e^x) + c}$$

$u = e^x$

$\frac{du}{dx} = e^x$

$du = e^x dx$

(ex) $$\int \frac{e^x}{e^x+15}\,dx = \int \frac{1}{u}\cdot du = \ln|u| + C$$

$$= \ln|e^x+15| + C$$

$$= \boxed{\ln(e^x+15)+C}$$

du

$u = e^x + 15$

$\frac{du}{dx} = e^x$

$du = e^x dx$

(ex) $$\int x \sec(x^2)\tan(x^2)\,dx = \int \frac{1}{2}\sec u \cdot \tan u \, du$$

$$= \frac{1}{2}\sec u + C = \boxed{\frac{1}{2}\sec(x^2)+C}$$

$u = x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

ex $\int \sin x \underbrace{\cos x\, dx}_{du} = \int u \cdot du = \frac{1}{2}u^2 + C$

$= \boxed{\frac{1}{2}(\sin x)^2 + C}$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\, dx$

ex $\int x^4(\underbrace{x^5+1})^8 dx = \int \frac{1}{5} \cdot u^8 \; du = \frac{1}{5} \cdot \frac{1}{9} u^9 + C$

$u = x^5+1$

$\frac{du}{dx} = 5x^4$

$du = 5x^4 dx$

$\frac{1}{5} du = x^4 dx$

$= \frac{1}{45} u^9 + C$

$= \boxed{\frac{1}{45}(x^5+1)^9 + C}$

ex) $\int \frac{s}{s-3}\,ds = \int \frac{u+3}{u}\,du = \int \frac{u}{u} + \frac{3}{u}\,du$

$= \int \left(1 + \frac{3}{u}\right) du = u + 3\ln|u| + C$

$= \boxed{s - 3 + 3\ln|s-3| + C}$

$u = s-3$

$du = ds$

$s = u+3$

ex) $\int \frac{\sec^2(\sqrt{x}+1)}{\sqrt{x}}\,dx = \int 2\sec^2 u\,du = 2\tan u + C$

$= \boxed{2\tan(\sqrt{x}+1) + C}$

$u = \sqrt{x}+1$

$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

ex $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int -\frac{1}{u} du$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$= -\ln|u| + C$

$= -\ln|\cos x| + C$

$= \ln|(\cos x)^{-1}| + C$

$= \ln\left|\frac{1}{\cos x}\right| + C$

$= \boxed{\ln|\sec x| + C}$ *memorize

ex $\int x^5\sqrt{x^3+1}\,dx = \int \underbrace{x^3}_{u-1}\underbrace{\sqrt{x^3+1}}_{\sqrt{u}}\underbrace{x^2dx}_{\frac{1}{3}du}$

$u = x^3+1 \rightarrow x^3 = u-1$

$\frac{du}{dx} = 3x^2$

$du = 3x^2dx$

$\frac{1}{3}du = x^2dx$

$= \int \frac{1}{3}(u-1)\sqrt{u}\,du$

$= \int \frac{1}{3}(u-1)u^{1/2}du$

$= \frac{1}{3}\int (u^{3/2} - u^{1/2})\,du$

$= \frac{1}{3}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C$

$= \frac{2}{15}u^{5/2} - \frac{2}{9}u^{3/2} + C$

$= \boxed{\frac{2}{15}(x^3+1)^{5/2} - \frac{2}{9}(x^3+1)^{3/2} + C}$

ex $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos x \, dx}{\sin^3 x} = \int_{1/\sqrt{2}}^{1} \frac{1}{u^3} \, du = \int_{1/\sqrt{2}}^{1} u^{-3} \, du =$

$u = \sin x$

$du = \cos x \, dx$

If $x = \pi/4$, $u = \sin(\pi/4) = 1/\sqrt{2}$

If $x = \pi/2$, $u = \sin(\pi/2) = 1$

$$\left. \frac{u^{-2}}{-2} \right|_{1/\sqrt{2}}^{1} =$$

$$\left(\frac{1^{-2}}{-2}\right) - \left(\frac{(1/\sqrt{2})^{-2}}{-2}\right)$$

$$= \frac{-1}{2} + \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^{-2}$$

$$= \frac{-1}{2} + \frac{1}{2}(\sqrt{2})^2 = \frac{-1}{2} + \frac{1}{2}(2) = 1 - 1/2$$

$$= 1/2$$

(ex) $$\int_0^2 \frac{2s}{s^2+1}\,ds = \int_1^5 \frac{1}{u}\,du = \ln|u|\Big|_1^5 = \ln 5 - \cancel{\ln 1}$$

$$= \boxed{\ln 5}$$

$u = s^2+1$

$\frac{du}{ds} = 2s$

$du = 2s\,ds$

If $s=0$, $u = 0^2+1 = 1$

If $s=2$, $u = 2^2+1 = 5$

(ex) $$\int_5^{10} \frac{8t+6}{2t^2+3t}\,dt = \int_{65}^{230} 2\,\frac{1}{u}\,du$$

$u = 2t^2+3t$

$\frac{du}{dt} = 4t+3$

$du = (4t+3)\,dt$

$2du = (8t+6)\,dt$

If $t=5$, $u = 2(5)^2+3\cdot 5 = 50+15 = 65$

if $t=10$, $u = 2\cdot 10^2 + 3\cdot 10 = 200+30 = 230$

$$= 2\ln|u|\Big|_{65}^{230} = 2\ln(230) - 2\ln(65)$$

$$= \boxed{2\ln\left(\frac{230}{65}\right)}$$

(ex) $\int e^{x+e^x}\,dx = \int \underbrace{e^x e^{e^x}\,dx} = \int du = u + C = \boxed{e^{e^x} + C}$

$u = e^{(e^x)}$

$\frac{dy}{dx} = e^{(e^x)} \cdot e^x$

$du = e^{e^x} e^x\,dx$

Check:

$\frac{d}{dx}\left\{ e^{e^x} + c \right\} = e^{e^x} \cdot e^x$ ✓

(ex) $\int \frac{dx}{e^x + e^{-x}} \left(\frac{e^x}{e^x}\right) = \int \frac{e^x \, dx}{(e^x)^2 + 1} = \int \frac{1}{u^2+1}\,du = \arctan u + C$

du

$= \boxed{\arctan(e^x) + C}$

$u = e^x$

$du = e^x\,dx$

Ch 7.2: Integration By Parts

(product rule – backwards)

$$\frac{d}{dx}\left\{ u(x)\cdot v(x) \right\} = u'(x)\,v(x) + u(x)\cdot v'(x)$$

So: $$\int \left[u'(x)v(x) + u(x)v'(x)\right] dx = u(x)\cdot v(x) + C$$

$$\int u'(x)v(x)\,dx + \underbrace{\int u(x)v'(x)\,dx} = u(x)\,v(x) + C$$

So: $$\int u(x)\cdot v'(x)\,dx = u(x)\,v(x) - \int u'(x)\,v(x)\,dx + C$$

Mnemonic: $$\boxed{\int u\,dv = uv - \int v\,du}$$ *memorize (ex) $\int x \sin x \, dx = -x\cos x - \int -\cos x (1)\, dx =$

$-x\cos x + \int \cos x \, dx$

$= \boxed{-x\cos x + \sin x + C}$

$u: x \qquad du: 1\,dx$

$dv: \sin x\, dx \qquad v: -\cos x$

$\int u\,dv = uv - \int v\,du$

(ex) $\int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\, dx$

$u: \ln x \qquad du: \frac{1}{x}\,dx$

$dv: x\,dx \qquad v: \frac{1}{2}x^2$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\, dx$

$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\left(\frac{1}{2}x^2\right) + C$

$= \boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

ex $\int (x+1)\sec^2 x\,dx = (x+1)\tan - \int \tan x \cdot dx$

$u: x+1$ $\quad du: 1\,dx$

$dv: \sec^2 x\,dx$ $\quad v: \tan x$

$$= (x+1)\tan x - \int \frac{\sin x}{\cos x}\,dx$$

$s = \cos x$

$ds = -\sin x\,dx$

$$= (x+1)\tan x - \int \frac{-ds}{s}$$

$$= (x+1)\tan x + \ln|s| + C$$

$$= \boxed{(x+1)\tan x + \ln|\cos x| + C}$$

ex) $\int \underbrace{x}\underbrace{e^{6x}}\, dx = \frac{x}{6}e^{6x} - \int \frac{1}{6}e^{6x}\,dx$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$u: x \qquad du: 1 \cdot dx$

$dv: e^{6x}dx \qquad v: \frac{1}{6}e^{6x}$

$= \frac{x}{6}e^{6x} - \frac{1}{6}\left(\frac{1}{6}e^{6x}\right) + C$

$= \boxed{\frac{1}{6}e^{6x}\left(x - \frac{1}{6}\right) + C}$

ex) $\int (3t+5)\cos\left(\frac{t}{4}\right)dt =$

$(3t+5)(4\sin(\frac{t}{4}))$

$- \int 12\sin(\frac{t}{4})dt$

$u: 3t+5 \qquad du: 3\,dt$

$dv: \cos(\frac{t}{4})dt \qquad v: 4\sin(\frac{t}{4})$

$= (3t+5)(4\sin(\frac{t}{4})) - 12\cdot(4)\left[-\cos(\frac{t}{4})\right] + C$

$= \boxed{4(3t+5)\sin\frac{t}{4} + 48\cos(\frac{t}{4}) + C}$

$v = \int dv = \int \cos(\frac{t}{4})dt = \int \cos(u)\cdot 4\,du$

$u = \frac{t}{4}$

$du = \frac{1}{4}dt$

$4du = dt$

$= 4\sin u$

$= \boxed{4\sin(\frac{t}{4})}$

(ex) $\int x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\, dx$

$u: \ln x \qquad du: \frac{1}{x}dx$

$dv: x^3 dx \qquad v: \frac{1}{4}x^4$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\left(\frac{1}{4}x^4\right) + C$

$= \boxed{\frac{1}{4}x^4\left(\ln x - \frac{1}{4}\right) + C}$

(ex) $\int x^2 \ln^2 x \, dx$

$u: \ln^2 x \qquad du: 2\ln x \cdot \frac{1}{x}\,dx$

$dv: x^2 dx \qquad v: \frac{1}{3}x^3$

$= \frac{1}{3}x^3 \ln^2 x - \int \frac{1}{3}x^3 \cdot 2\ln x \cdot \frac{1}{x}\,dx$

$= \frac{1}{3}x^3 \ln^2 x - \frac{2}{3}\int x^2 \ln x \, dx =$

$u: \ln x \qquad du = \frac{1}{x}dx$

$dv: x^2 dx \qquad v = \frac{1}{3}x^3$

$\frac{1}{3}x^3 \ln^2 x - \frac{2}{3}\left[\frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\,dx\right]$

$= \frac{1}{3}x^3 \ln^2 x - \frac{2}{3}\left[\frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx\right]$

$= \frac{1}{3}x^3 \ln^2 x - \frac{2}{3}\left[\frac{1}{3}x^3 \ln x - \frac{1}{3}\cdot\frac{1}{3}x^3\right] + C$

ex $\int \ln x \, dx = x\ln x - \int x\left(\frac{1}{x}\right)dx$

$u: \ln x \qquad du: \frac{1}{x}dx$

$= x\ln x - \int 1 \, dx$

$= \boxed{x\ln x - x + C}$

$dv: 1\,dx \qquad v: x$

ex $\int \arctan x \, dx = x\arctan x - \int \frac{x}{1+x^2}dx =$

$\frac{1}{2}dw$

$u: \arctan x \qquad du: \frac{1}{1+x^2}dx$

$dv: 1 \cdot dx \qquad v = x$

Substitution:

$w = 1+x^2$

$\frac{dw}{dx} = 2x$

$dw = 2x\,dx$

$\frac{1}{2}dw = x\,dx$

$x\arctan x - \int \frac{1}{2}\frac{1}{w}\,dw$

$= x\arctan x - \frac{1}{2}\ln|w| + C$

$= x\arctan x - \frac{1}{2}\ln|1+x^2| + C$

$= \boxed{x\arctan - \frac{1}{2}\ln(1+x^2) + C}$

(ex) $\int \arcsin x \, dx$

Recall:

$$\frac{d}{dx}\{\arcsin x\} = \frac{1}{\sqrt{1-x^2}}$$

$$\int u\,dv = uv - \int v\,du$$

$u : \arcsin x \quad du : \frac{1}{\sqrt{1-x^2}}\,dx$

$dv : dx \quad v : x$

$$= x\arcsin x - \int \frac{x}{\underbrace{\sqrt{1-x^2}}}\,dx \quad -\tfrac{1}{2}\,dw$$

$w = 1-x^2$

$dw = -2x\,dx$

$-\frac{1}{2}\,dw = x\,dx$

$$= x\arcsin x - \int -\frac{1}{2}\frac{1}{\sqrt{w}}\,dw = x\arcsin x + \frac{1}{2}\int w^{-1/2}\,dw$$

$$= x\arcsin x + \frac{1}{2}\cdot(2)\,w^{1/2} + C$$

$$= x\arcsin x + \sqrt{w} + C$$

$$= \boxed{x\arcsin x + \sqrt{1-x^2} + C}$$

" Integrating Around in a Circle "

$$\int u\,dv = uv - \int v\,du$$

ex) $\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$

$u: e^x$ $\quad du: e^x\,dx$

$dv: \cos x\,dx$ $\quad v: \sin x$

$u: e^x$ $\quad du: e^x\,dx$

$dv: \sin x\,dx$ $\quad v: -\cos x$

$$= e^x \sin x + \left[+ e^x \cos x + \int - e^x \cos x\,dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$$

$+\int e^x \cos x\,dx$ to both sides

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$

$$\boxed{\int e^x \cos x\,dx = \frac{1}{2}\left[e^x \sin x + e^x \cos x\right] + C}$$

Integration by Parts:

$$\int u\,dv = uv - \int v\,du$$

(ex) $\boxed{\int e^x \sin x\,dx} =$

$u : e^x$ $\qquad$ $du = e^x dx$

$du : \sin x\,dx$ $\qquad$ $v = -\cos x$

$$-e^x\cos x - \int -e^x \cos x\,dx$$

$$= -e^x \cos x + \int e^x \cos x\,dx$$

$u : e^x$ $\qquad$ $du = e^x dx$

$dv : \cos x\,dx$ $\qquad$ $v = \sin x$

$$= \boxed{-e^x\cos x + e^x \sin x - \int e^x \sin x\,dx}$$

$+\int e^x \sin x\,dx$ to both sides

$$2\int e^x \sin x\,dx = -e^x \cos x + e^x \sin x + C$$

$$\boxed{\int e^x \sin x\,dx = \frac{1}{2}\left[-e^x\cos x + e^x \sin x\right] + C}$$

(ex) $\int_0^{10} xe^x\,dx = xe^x\Big|_0^{10} - \int_0^{10} e^x\,dx$

$u: x \qquad du: 1\,dx$

$dv: e^x dx \qquad v: e^x$

$= (10e^{10} - 0) - \int_0^{10} e^x\,dx$

$= 10e^{10} - [e^{10} - e^0]$

$= 10e^{10} - e^{10} + e^0$

$= \boxed{9e^{10} + 1}$ Area under curve

(ex) $\int_0^{2\pi} x^2 \sin x\,dx = -x^2\cos x\Big|_0^{2\pi} - \int_0^{2\pi} -2x\cos x\,dx$

$u: x^2 \qquad du: 2x\,dx$

$dv: \sin x\,dx \quad v: -\cos x$

$= -(2\pi)^2\cos(2\pi) - (-0)\cos 0 + \int_0^{2\pi} 2x\cos x\,dx$

$= -4\pi^2 + \int_0^{2\pi} 2x\cos x\,dx$

$u: 2x \qquad du = 2\,dx$

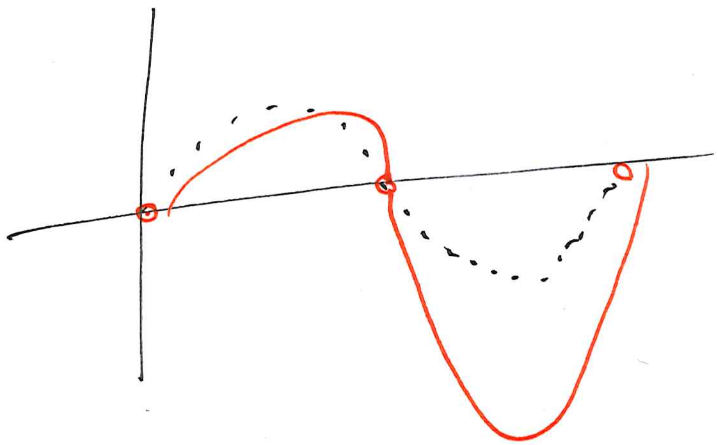
$dv: \cos x\,dx \quad v: \sin x$

$= -4\pi^2 + 2x\sin x\Big|_0^{2\pi} - \int_0^{2\pi} 2\sin x\,dx$

$= -4\pi^2 + (4\pi\underbrace{\cancel{\sin(2\pi)}}_{0} - 0) - \int_0^{2\pi} 2\sin x\,dx$

$= -4\pi^2 - 2\left[-\cos x\Big|_0^{2\pi}\right] = -4\pi^2 - 2(-1-(-1)) = \boxed{-4\pi^2}$

$x^2 \sin x$:



net area should be negative

$\boxed{\text{Trig Identities}}$

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$\cos^2 x = \frac{1 + \cos 2x}{2}$

Ch 7.3 : Trig Integrals

Integrating functions of the form $\sin^a x \cos^b x$ and $\sec^a x \tan^b x$

Recall: (1) $\sin^2 x + \cos^2 x = 1$

(2) $\tan^2 x + 1 = \sec^2 x$

(3) $\sin^2 x = \dfrac{1-\cos(2x)}{2}$

(4) $\cos^2 x = \dfrac{1+\cos(2x)}{2}$

Remark: If you forget (2), you can get it from (1):

$$\frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

(ex) $\displaystyle\int \underbrace{\sin^{10} x}_{u^{10}} \underbrace{\cos x\, dx}_{du} = \int u^{10}\, du = \frac{1}{11} u^{11} + C = \boxed{\frac{1}{11}\sin^{11} x + C}$

$u = \sin x$

$\dfrac{du}{dx} = \cos x$

$du = \cos x\, dx$

ex

$$\int \sin^{10}x \cdot \cos^5 x\,dx = \int \underbrace{\sin^{10}x}_{u^{10}} \cdot \underbrace{\cos^4 x}_{?} \cdot \underbrace{\cos x\,dx}_{du}$$

need to change to sines

Idea: $u = \sin x$
$du = \cos x\,dx$

Identity: $\sin^2 x + \cos^2 x = 1$

so $\cos^2 x = 1 - \sin^2 x$

$\cos^4 x = (\cos^2 x)^2 = (1-\sin^2 x)^2$

$$= \int \underbrace{\sin^{10}x}_{u^{10}} \cdot \underbrace{(1-\sin^2 x)^2}_{(1-u^2)^2} \underbrace{\cos x\,dx}_{du} = \int u^{10}(1-u^2)^2\,du$$

$$= \int u^{10}(1-2u^2+u^4)\,du = \int (u^{10} - 2u^{12} + u^{14})\,du = \frac{1}{11}u^{11} - 2\frac{1}{13}u^{13} + \frac{1}{15}u^{15} + C$$

$$= \boxed{\frac{1}{11}\sin^{11}x - \frac{2}{13}\sin^{13}x + \frac{1}{15}\sin^{15}x + C}$$

ex $\int \sin^5 x \cos^4 x \, dx$

Idea: $\int \sin^5 x \cdot (\cos^2 x)^2 dx$

$= \int \sin^5 x (1-\sin^2 x)^2 dx$

only sine: $u = \sin x$, $du = \cos x\, dx \leftarrow ??$

Another idea:

$\int \sin^4 x \cos^4 x \sin x \, dx$

$= \int (\sin^2 x)^2 \cos^4 x \sin x \, dx$

$= \int (1-\cos^2 x)^2 \cos^4 x \underbrace{\sin x \, dx}_{-du}$

$u = \cos x$
$du = -\sin x \, dx$
$-du = \sin x \, dx$

$-\int (1-u^2)^2 u^4 \, du = -\int (1-2u^2+u^4) u^4 \, du = -\int u^4 - 2u^6 + u^8 \, du$

$= -\left[\frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C\right] = \frac{-1}{5}u^5 + \frac{2}{7}u^7 - \frac{1}{9}u^9 + C$

$= \boxed{\frac{-1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + C}$

General Idea:

$$\int \sin^a x \cos^b x \, dx :$$

If a power is odd, reserve one of them as dx

So, u: use other

ex $\int \sin^{17} x \cos^{16} x \, dx = \int \underbrace{\sin^{16} x}_{\text{convert to cosine}} \cos^{16} x \underbrace{\sin x \, dx}_{-du}$

so: $u = \cos x$

ex

$$\int \sin^{2.71} x \cdot \cos^3 x \, dx = \int \sin^{2.71} x \cdot \cos^2 x \cdot \underbrace{\cos x \, dx}_{du}$$

$u : \sin x$

$$= \int \sin^{2.71} x \cdot (1 - \sin^2 x) \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int u^{2.71} (1 - u^2) \, du = \int \left(u^{2.71} - u^{4.71} \right) du$$

$$= \frac{u^{3.71}}{3.71} - \frac{u^{5.71}}{5.71} + C$$

$$= \boxed{\frac{(\sin x)^{3.71}}{3.71} - \frac{(\sin x)^{5.71}}{5.71} + C}$$

ex $\int \sin^5 x\, dx = \int \sin^4 x \underbrace{\sin x\, dx}_{\text{"}du\text{"}}$

$u: \cos x$

$= \int (\sin^2 x)^2 \sin x dx = \int (1-\cos^2 x)^2 \sin x\, dx$

$u = \cos x$

$du = -\sin x\, dx$

$-du = \sin x\, dx$

$= \int (1-u^2)^2 (-1)\, du$ $\boxed{\text{etc}}$

What if powers are even?

ex $\int \sin^2 x\,dx$

use half-angle formulas:

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$\int \sin^2 x\,dx = \int \frac{1-\cos(2x)}{2}\,dx = \frac{1}{2}\int 1-\cos(2x)\,dx$$

$u = 2x$
$du = 2\,dx$
$\frac{1}{2}du = dx$

$$= \frac{1}{2}\int [1-\cos(u)]\,\frac{1}{2}\cdot du$$

$$= \frac{1}{4}\int 1-\cos u\,du = \frac{1}{4}[u-\sin u] + C$$

$$= \boxed{\frac{1}{4}[2x-\sin(2x)]+C}$$

ex $\int \sin^2 x \cos^2 x \, dx$

Recall:

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)\, dx = \frac{1}{4}\int \left[1 - \cos^2(2x)\right] dx$$

$$\cos^2(2x) = \frac{1+\cos(4x)}{2}$$

$$= \frac{1}{4}\int \left[1 - \frac{1+\cos(4x)}{2}\right] dx$$

$$= \frac{1}{4}\int \left(1 - \frac{1}{2} - \frac{1}{2}\cos(4x)\right) dx$$

$$= \frac{1}{4}\int \left(\frac{1}{2} - \frac{1}{2}\cos(4x)\right) dx$$

$$= \boxed{\frac{1}{4}\left[\frac{1}{2}x - \frac{1}{8}\sin(4x)\right] + C}$$

Idea:

last ex

$$\frac{1}{2}\int \cos^2 u \, du = \frac{1}{2}\int (1-\sin^2 u)\, du$$

$\cancel{u = \sin u}$

$\cancel{du = \cos u \, du}$???

Products of Secants and Tangents

Products of Secants and Tangents

$$\int \tan x\,dx = \ln|\sec x| + C$$

The antiderivative of the tangent function is the natural log of the absolute value of the secant function, plus any constant

$$\int \sec x\,dx = \int \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx = \int \frac{\overbrace{\sec^2 x + \sec x \tan x\,dx}^{du}}{\underbrace{\sec x + \tan x}_{u}}$$

$u = \sec x + \tan x$

$\frac{du}{dx} = \sec x \tan x + \sec^2 x$

$du = [\sec^2 x + \sec x \tan x]\,dx$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \boxed{\ln|\sec x + \tan x| + C}$$ memorize

ex) $\int \sec^2 x \tan x \, dx$

1 way: $u = \tan x$, $du = \sec^2 x \, dx$ $\quad \int u \, du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\tan^2 x + C}$

Reserve $\boxed{2}$ secants (not one)

secant: even power

Another way: $u = \sec x$, $du = \sec x \tan x \, dx$ $\quad = \boxed{\frac{1}{2}\sec^2 x + C}$

$$\int \sec x \cdot \underbrace{\sec x \tan x \, dx}_{du} = \int u \, du = \frac{1}{2}u^2 + C$$

Note: $\boxed{\frac{1}{2}\tan^2 x} + C = \frac{1}{2}(\sec^2 x - 1) + C = \boxed{\frac{1}{2}\sec^2 x - \frac{1}{2}} + C$

$= \frac{1}{2}\sec^2 x + C$ arbitrary constant

Reserve: $\sec x \tan x$ as dx

Odd power of tangent

$u = \sec x$

Last Time:

$\int \sec^m x \cdot \tan^n x \, dx$:

Even power of secant:
Reserve $\sec^2 x$ for du
Other secants → tangents
$u = \tan x$

Odd power of tangent:
Reserve $\sec x \tan x$ for du
Other tangents → secants
$u = \sec x$

Odd power of secant, even power of tangent: reduction formula (we'll skip this)

IDENTITY:
$\tan^2 x + 1 = \sec^2 x$
so
$\tan^2 x = \sec^2 x - 1$

ex $\int \sec^3 x \tan^3 x \, dx = \int \sec^2 x \cdot \tan^2 x \cdot \sec x \tan x \, dx$

$= \int \sec^2 x \, (\sec^2 x - 1) \cdot \underbrace{\sec x \tan x \, dx}_{du} = \int u^2 (u^2 - 1) \, du$

$u = \sec x$
$du = \sec x \tan x \, dx$

$= \int (u^4 - u^2) \, du = \frac{u^5}{5} - \frac{u^3}{3} + C = \boxed{\frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C}$

→ Able to do suggested problems through 7.3

Ch. 7.4 Trig Substitution

Motivation:

$$\int_3^7 \frac{1}{\sqrt{x^2+2x+1}}\,dx = \int_3^7 \frac{1}{\cancel{\sqrt{(x+1)^2}}}\,dx$$

$$= \int_3^7 \frac{1}{x+1}\,dx = \ln|x+1| \Big|_3^7$$

$$= \ln 8 - \ln 4 = \ln(8/4) = \boxed{\ln 2}$$

Nice thing:

$\cancel{\sqrt{(\ )^2}}$

get rid of $\sqrt{\ }$

Very similar integrand

function we're integrating

Goal:

$\sqrt{x^2+1} = \cancel{\sqrt{(\quad)^{\cancel{2}}}}$

cancel $\sqrt{\ }$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx$$

Recall: $(\tan\theta)^2+1 = (\sec\theta)^2$

So, if $x=\tan\theta$: $x^2+1 = \tan^2\theta+1 = (\sec\theta)^2$

then: $\sqrt{x^2+1} = \cancel{\sqrt{(\sec\theta)^{\cancel{2}}}} = \sec\theta$

Sub: $x=\tan\theta$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sec\theta}\cdot \sec^2\theta\, d\theta = \int \sec\theta\, d\theta$$

(last time – memorize)

$$= \ln\left|\underbrace{\sec\theta}_{\sqrt{x^2+1}} + \underbrace{\tan\theta}_{x}\right| + C$$

$\theta \to x$

$$= \boxed{\ln\left|\sqrt{x^2+1} + x\right| + C}$$

Sub: $x = \tan\theta$

what is $\sec\theta$?

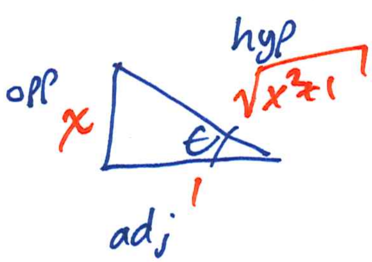
$\boxed{\text{2 WAYS}}$

- 1^{ST} Calc: $\sqrt{x^2+1} = \sec\theta$

- Draw a triangle:

$x = \tan\theta$

$\frac{x}{1} = \frac{opp}{adj}$



Then: $\sec\theta = \frac{hyp}{adj} = \frac{\sqrt{x^2+1}}{1}$

So $\sec\theta = \sqrt{x^2+1}$

Idea: $\int \frac{1}{\sqrt{1+x^2}}\,dx = \int (1+x^2)^{-1/2}\,dx = \dots ?$

---

ex $\int x\sqrt{9-x^2}\,dx$

Note: Easier to solve using $u = 9-x^2$

To practice the method, we'll use trig sub

Form: $\sqrt{(\text{quadratic})} \xrightarrow[\text{goal}]{} \cancel{\sqrt{(\quad)^2}}$ get rid of $\sqrt{\ }$

Choose substitution

Identities:

$1-\sin^2\theta = \cos^2\theta$

$1+\tan^2\theta = \sec^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

const − fcn

const + fcn

fcn − const

have $9 - x^2$

const − fcn

Closest identity

$1 - \sin^2\theta = \cos^2\theta$

fix const

want $9 - 9\sin^2\theta = 9\cos^2\theta$

Need: $x^2 = 9\sin^2\theta$

Use: $\boxed{x = 3\sin\theta}$

Check that it's a good idea by looking ahead

($\sqrt{\ } \to$ cancel !)

$$9-x^2 = 9-(3\sin\theta)^2$$
$$= 9-9\sin^2\theta$$
$$= 9(1-\sin^2\theta)$$
$$= 9\cos^2\theta$$

so: $\sqrt{9-x^2} = \cancel{\sqrt{9\cos^2\theta}}$
$= 3\cos\theta$

$\sqrt{\ }$ went away – good substitution!

Do substitution: $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$

$$\int x\sqrt{9-x^2}\,dx = \int 3\sin\theta \cdot 3\cos\theta \cdot 3\cos\theta\, d\theta = \int 27\cdot\cos^2\theta\sin\theta\, d\theta$$

$u = \cos\theta$

$-du = \sin\theta\, d\theta$

Evaluate:

$$-27\int u^2\,du = -27\cdot\tfrac{1}{3}u^3 + C$$

Get original variable back

$$-9u^3 + C \underset{\underset{u=\cos\theta}{\uparrow}}{=} -9 \cdot \cos^3\theta + C = -9 \cdot \left(\tfrac{1}{3}\sqrt{9-x^2}\right)^3 + C$$

Already did:

$\sqrt{9-x^2} = 3\cos\theta$

so $\frac{1}{3}\sqrt{9-x^2} = \cos\theta$

$$= \frac{-9}{27}(9-x^2)^{3/2} + C = \boxed{\frac{-1}{3}(9-x^2)^{3/2} + C}$$

$$\sqrt{A}^3 = (A^{1/2})^3 = A^{3/2}$$

ex $\int \frac{1}{(x^2-16)^{3/2}}\,dx = \int \frac{1}{\sqrt{x^2-16}^{\,3}}\,dx$

want $\sqrt{\ }$ go away

$1-\sin^2\theta = \cos^2\theta$ const - fcn

$1+\tan^2\theta = \sec^2\theta$ const + fcn

$\sec^2\theta - 1 = \tan^2\theta$ fcn - const ← closest

have: x^2-16

fcn - const

$\sec^2\theta - 1 = \tan^2\theta$

fix constant

want: $16\sec^2\theta - 16 = 16\tan^2\theta$

$x^2 = 16\sec^2\theta$

use $\boxed{x = 4\sec\theta}$ as substitution

Check that $x = 4\sec\theta$ really gets rid of $\sqrt{\ }$

$$x^2 - 16 = (4\sec\theta)^2 - 16$$
$$= 16\sec^2\theta - 16$$
$$= 16(\sec^2\theta - 1)$$
$$= 16 \cdot \tan^2\theta$$

So: $\sqrt{x^2-16} = \sqrt{16 \cdot \tan^2\theta}$
$= 4\tan\theta$

$\sqrt{\ }$ went away — good substitution!

Do substitution: $x = 4\sec\theta$
$dx = 4\sec\theta \tan\theta \, d\theta$

$$\int \frac{1}{\sqrt{x^2-16}^3}\,dx = \int \frac{1}{(4\tan\theta)^3} 4\sec\theta\tan\theta\,d\theta = \frac{1}{16}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

Evaluate $\frac{1}{16}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{16}\int \frac{1}{\cos\theta}\cdot\left(\frac{\cos\theta}{\sin\theta}\right)^2 d\theta$

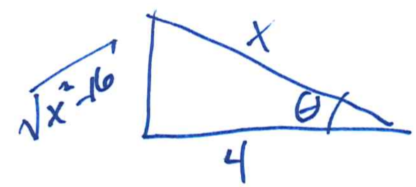
$$= \frac{1}{16}\int \frac{\overbrace{\cos\theta \quad d\theta}^{du}}{\sin^2\theta} \qquad = \frac{1}{16}\int \frac{1}{u^2}\,du = \int u^{-2}\,du$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

Used: $x = 4\sec\theta$

$\frac{hyp}{adj} = \sec\theta = \frac{x}{4}$



$\sin\theta = \frac{opp}{hyp} = \frac{\sqrt{x^2-16}}{x}$

$$= -\frac{1}{16}u^{-1} + C$$

Get original variable back

$$= \frac{-1}{16}\cdot(\sin\theta)^{-1} + C = \frac{-1}{16\sin\theta} + C$$

$$= \boxed{\frac{-1}{16}\cdot\frac{x}{\sqrt{x^2-16}} + C}$$

(ex) $\int \frac{\sqrt{4x^2-1}}{x}\,dx$

Identities:

$1-\sin^2\theta = \cos^2\theta$

$1+\tan^2\theta = \sec^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$4x^2 - 1$

fcn - const

$\sec^2\theta - 1 = \tan^2\theta$

Need: $4x^2 = \sec^2\theta$

$\boxed{2x = \sec\theta}$

Check substitution:

$4x^2 - 1 = (2x)^2 - 1 = (\sec\theta)^2 - 1 = \tan^2\theta$

So: $\sqrt{4x^2-1} = \cancel{\sqrt{\tan^2\theta}} = \tan\theta$

$\sqrt{\ }$ cancelled: good sub!

Do subst:

$2x = \sec\theta$

$x = 1/2 \sec\theta$

$dx = \frac{1}{2}\sec\theta\tan\theta\, d\theta$

$$\int \frac{\sqrt{4x^2-1}}{x}\,dx$$

$$= \int \frac{\tan\theta}{\cancel{\frac{1}{2}\sec\theta}}\,\cancel{\frac{1}{2}\sec\theta}\,\tan\theta\, d\theta = \int \tan^2\theta\, d\theta$$

$$= \int (\sec^2\theta - 1)\,d\theta = \tan\theta - \theta + C$$

used:

$x = \frac{1}{2}\sec\theta$

$2x = \sec\theta$

$\operatorname{arcsec}(2x) = \theta$

$\tan\theta = \sqrt{4x^2-1}$

Need to get x back:

$$= \boxed{\sqrt{4x^2-1} - \operatorname{arcsec}(2x) + C}$$

Completing the Square

ex $\displaystyle\int \frac{1}{\sqrt{3-x^2+2x}}\,dx$

3 pieces

$\downarrow$ complete $\square$

2 pieces

$\downarrow$ trig

1 piece $\cancel{\sqrt{(\ \ )^2}}$

$3-x^2+2x =$

$-[\underbrace{x^2-2x}-3]$

$= -[\underbrace{x^2-2x+1}\underbrace{-1-3}]$

$= -[(x-1)^2-4]$

$= 4-(x-1)^2$

Recall:

$(x+a)^2 = \underbrace{x^2+2ax}+a^2$

$a=-1$

$(x-1)^2 = x^2-2x+1$

Choose sub:

const − fcn

$1 - \sin^2\theta = \cos^2\theta$

have: $4 - \underbrace{(x-1)^2}$

$1 - \sin^2\theta = \cos^2\theta$

match constants $4 - \underbrace{4\sin^2\theta} = 4\cos^2\theta$

Need: $(x-1)^2 = 4\sin^2\theta$

$\boxed{x - 1 = 2\sin\theta}$

Also can say ~~[illegible]~~

$\boxed{x = 1 + 2\sin\theta}$

Check our sub:

$$x = 1 + 2\sin\theta$$
$$dx = 2\cos\theta\, d\theta$$

$$\begin{aligned} 3 - x^2 + 2x &= 4 - (x-1)^2 \\ &= 4 - (2\sin\theta)^2 \\ &= 4 - 4\sin^2\theta \\ &= 4(1 - \sin^2\theta) \\ &= 4\cos^2\theta \end{aligned}$$

Then: $\sqrt{3 - x^2 + 2x} = \sqrt{4\cos^2\theta} = 2\cos\theta$ $\sqrt{\ }$ gone!

$$\int \frac{1}{\sqrt{3 - x^2 + 2x}}\, dx = \int \frac{1}{2\cos\theta} \cdot 2\cos\theta\, d\theta = \int 1\, d\theta = \theta + C$$

$$= \boxed{\arcsin\left(\frac{x-1}{2}\right) + C}$$

$$x - 1 = 2\sin\theta$$
$$\frac{x-1}{2} = \sin\theta$$
$$\theta = \arcsin\left(\frac{x-1}{2}\right)$$

→ Suggested Prob: $\boxed{§7.4}$

Feb 28

Ch 7.5 : Partial Fractions

Motivation: Fact: $\frac{1}{x+1} - \frac{1}{2x-1} = \frac{x-2}{(x+1)(2x-1)}$

$\int \frac{1}{x+1} - \frac{1}{2x-1}\,dx$: easy enough

$\underbrace{\int \frac{1}{x+1}\,dx}_{u=x+1} - \underbrace{\int \frac{1}{2x-1}\,dx}_{u=2x-1}$ etc

$\int \frac{x-2}{(x+1)(2x-1)}\,dx$: pretty tough

Method of Partial Fractions:

re-write a rational function as a sum of rational functions that are easy to integrate.

$\frac{\text{polynomial}}{\text{polynomial}}$

(just algebra!)

1st case: Denominator — Repeated Linear Factors

$$\frac{\text{numerator}}{(ax+r)^n} = \frac{c_1}{ax+r} + \frac{c_2}{(ax+r)^2} + \frac{c_3}{(ax+r)^3} + \dots + \frac{c_n}{(ax+r)^n}$$

numerator: polynomial, degree $< n$

a, r const

n natural

ex $\displaystyle\int \frac{6x+7}{\underbrace{4x^2+20x+25}_{\text{rational}}}\,dx$

denominator: $(2x+5)^2$

$$\frac{6x+7}{(2x+5)^2} = \underbrace{\frac{C}{(2x+5)} + \frac{D}{(2x+5)^2}}_{\text{easier to } \int}$$

Find C, D

$$= \frac{C(2x+5)+D}{(2x+5)^2}$$

common denominator: $(2x+5)^2$

$$\underbrace{6}x+\underbrace{7} = C(2x+5)+D = \underbrace{(2C)}x + \underbrace{(5C+D)}$$

$6 = 2C \longrightarrow \boxed{C=3}$

and $7 = 5C+D$

$7 = 5\cdot 3+D$

$= 15+D$

so $\boxed{D=-8}$

$$\int \frac{6x+7}{(2x+5)^2}\,dx = \int \frac{3}{2x+5} + \frac{-8}{(2x+5)^2}\,\underbrace{dx} = \int \left[\frac{3}{u} - \frac{8}{u^2}\right]\underbrace{\cdot \frac{1}{2} \cdot du}$$

$u = 2x+5$

$du = 2dx$

$\frac{1}{2}du = dx$

$$= \frac{1}{2}\int \frac{3}{u} - 8u^{-2}\,du = \frac{1}{2}\left[3\ln|u| + 8u^{-1}\right] + C$$

$$= \boxed{\frac{1}{2}\left[3\ln|2x+5| + \frac{8}{2x+5}\right] + C}$$

ex $\int \frac{x^2+6x+10}{(x+3)^3}\,dx$

$$\frac{x^2+6x+10}{(x+3)^3} = \frac{C}{(x+3)} + \frac{D}{(x+3)^2} + \frac{E}{(x+3)^3}$$

Find C, D, E

$$= \frac{C(x+3)^2 + D(x+3) + E}{(x+3)^3}$$

Common Denominator $(x+3)^3$

$$x^2+6x+10 = C(x+3)^2 + D(x+3) + E$$

if $x=-3$:

$$9-18+10 = 0+0+E$$

$\boxed{1=E}$

$\boxed{C=1}$

$$x^2+6x+10 = C(x^2+6x+9) + D(x+3) + 1$$

$$= x^2(C) + x(6C+D) + (9C+3D+1)$$

$$= x^2 + x(6+D) + (10+3D)$$

$6+D=6$

$\boxed{D=0}$

$$\int \frac{x^2+6x+10}{(x+3)^3}\,dx = \int \frac{1}{x+3} + \frac{1}{(x+3)^3}\,dx = \text{etc.}$$

Case 2: Denom has distinct linear factors

all differant

no 2 same

Rule:
$$\frac{\text{num}}{(a_1x+r_1)(a_2x+r_2)\cdots(a_nx+r_n)} = \frac{A}{(a_1x+r_1)} + \frac{B}{(a_2x+r_2)} + \cdots + \frac{C}{(a_nx+r_n)}$$

a_i, r_i const

num: polynomial, degree $< n$

a_ix+r_i all differant

ex $\displaystyle\int \frac{7x+13}{2x^2+x-10}\,dx$

$$\frac{7x+13}{(2x+5)(x-2)} = \frac{A}{2x+5} + \frac{B}{x-2}$$ Find A, B

$$= \frac{A(x-2) + B(2x+5)}{(2x+5)(x-2)}$$ common denom

$$\underbrace{7}x+\underbrace{13} = Ax - 2A + 2Bx + 5B$$
$$= x\underbrace{(A+2B)} + \underbrace{(-2A+5B)}$$

$7 = A + 2B \rightarrow A = 7 - 2B$

$13 = -2A + 5B$

$13 = -2(7-2B) + 5B$

$13 = -14 + 4B + 5B$

$27 = 9B$

$\boxed{B=3}$

$A = 7 - 2(3)$

$\boxed{A=1}$

$$\int \frac{7x+13}{(2x+5)(x-2)}\,dx = \underbrace{\int \frac{1}{2x+5} + \frac{3}{x-2}\,dx}_{\text{easier}}$$

Case 3: Some distinct, some repeated linear factors in denom

ex $$\frac{4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \quad \text{etc.}$$

Possible Complication: deg of num $\geq$ deg of denom

In this case: DIVIDE

ex: $\frac{13}{3} = \underbrace{\frac{12+1}{3} = \frac{4 \cdot 3 + 1}{3}}_{\text{pulled out biggest multiple of denom from num}} = \underbrace{\frac{4 \cdot \cancel{3}}{\cancel{3}} + \frac{1}{3}}_{\text{separate fraction + cancel}} = 4 + \frac{1}{3}$

ex: $\frac{x^2 + 4x - 7}{x^2 - 3x + 2} = \frac{x^2 - 3x + 2 \quad + 7x - 9}{x^2 - 3x + 2} = \frac{x^2 - 3x + 2}{x^2 - 3x + 2} + \frac{7x - 9}{x^2 - 3x + 2}$

deg of num = deg of denom so partial fractions won't work (yet)

$= 1 + \underbrace{\frac{7x-9}{x^2-3x+2}}_{\text{deg of num < deg of denom: can do partial fractions}} = 1 + \frac{7x-9}{(x-1)(x-2)}$ $\boxed{\text{etc...}}$

ex

$$\frac{x^3 - 3x^2 + 9x - 9}{x^2 - 3x + 2} = \frac{x^3 - 3x^2 + 2x \quad + 7x - 9}{x^2 - 3x + 2}$$

Can't do partial fractions yet - deg of num too big

$$= \frac{x^3 - 3x^2 + 2x}{x^2 - 3x + 2} + \frac{7x - 9}{x^2 - 3x + 2}$$

Note: $x(x^2 - 3x + 2) = x^3 - 3x^2 + 2x$

$$= \frac{x\cancel{(x^2 - 3x + 2)}}{\cancel{x^2 - 3x + 2}} + \frac{7x - 9}{x^2 - 3x + 2}$$

$$= \underbrace{x + \frac{7x - 9}{x^2 - 3x + 2}}_{\text{now: can do partial fractions}}$$

ex $$\frac{2x^3-5x^2+8x-7}{x^2-3x+2} = \frac{2x^3-6x^2+4x+x^2+4x-7}{x^2-3x+2}$$

Note: $2x(x^2-3x+2)$
$= 2x^3-6x^2+4x$

$$= \frac{2x\cancel{(x^2-3x+2)}}{\cancel{x^2-3x+2}} + \frac{x^2+4x-7}{x^2-3x+2}$$

$$= 2x + \frac{x^2+4x-7}{x^2-3x+2} = 2x + \frac{x^2-3x+2+7x-9}{x^2-3x+2}$$

Note: x^2-3x+2

$$= 2x + \frac{x^2-3x+2}{x^2-3x+2} + \frac{7x-9}{x^2-3x+2}$$

$$= 2x+1+\underbrace{\frac{7x-9}{x^2-3x+2}}_{\text{can do part frac}}$$

All suggested HW through 7.5

Ch 7.7: Numerical Integration

Motivation: sometimes we can't (don't want to) find antiderivative

Recall: $\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$

ex $\int e^{x^2}\,dx$

ex $\int \frac{1}{\ln x}\,dx$

ex $\int \sin(x^2)\,dx$

Absolute vs Relative Error

Absolute Error:

$|\text{exact} - \text{approx}|$

Relative Error:

$\dfrac{\text{abs error}}{|\text{actual}|}$

Case 1: 500g sack of flour mistakenly labeled 495g

$|500 - 495| = 5g$

$\dfrac{5}{500} = \dfrac{1}{100} = 1\%$

Case 2: 5g bottle of medicine mistakenly labeled 10g

$|5 - 10| = 5g$

$\dfrac{5}{5} = 1 = 100\%$

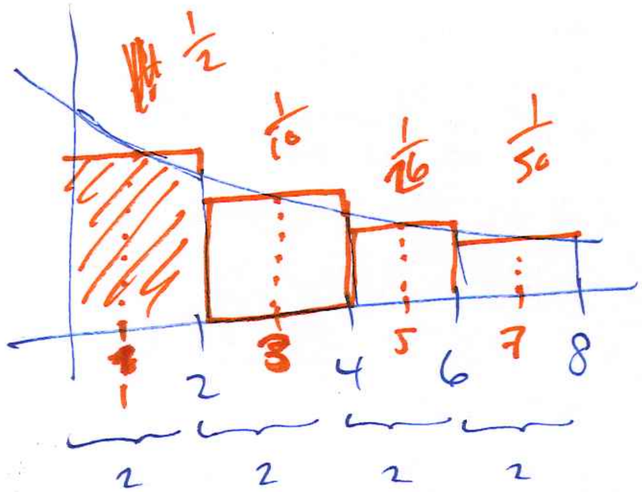
ex) We already saw midpt Riemann Sums

If we take n intervals (~~not~~ limit)

"Midpt Approximation"

ex) approx $\int_0^8 \frac{1}{1+x^2}\,dx$ using midpt approx, $n=4$

$$\approx 2\left(\tfrac{1}{2}\right) + 2\left(\tfrac{1}{10}\right) + 2\left(\tfrac{1}{26}\right) + 2\left(\tfrac{1}{50}\right)$$

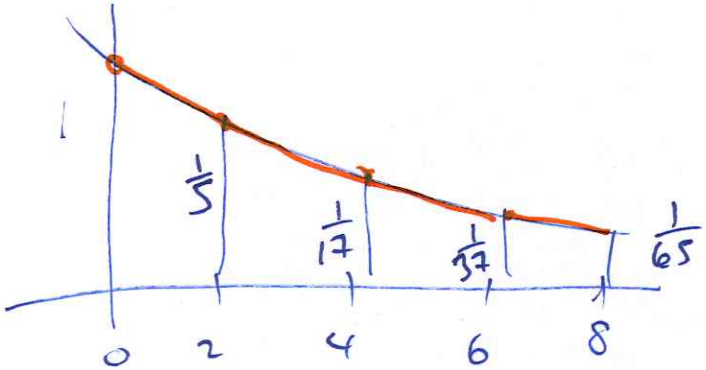


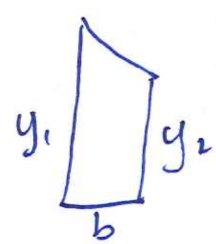
Approx fcn by constant line

ex) $\int_0^8 \frac{1}{1+x^2}\,dx$

Approx fcn using lines

"Trapezoid Rule"





Area of trapezoid:

$\frac{1}{2}b(y_1+y_2)$

$$\frac{1}{2}(2)\left(1+\frac{1}{5}\right) + \frac{1}{2}(2)\left(\frac{1}{5}+\frac{1}{17}\right) + \frac{1}{2}(2)\left(\frac{1}{17}+\frac{1}{37}\right) + \frac{1}{2}(2)\left(\frac{1}{37}+\frac{1}{65}\right)$$

General Form

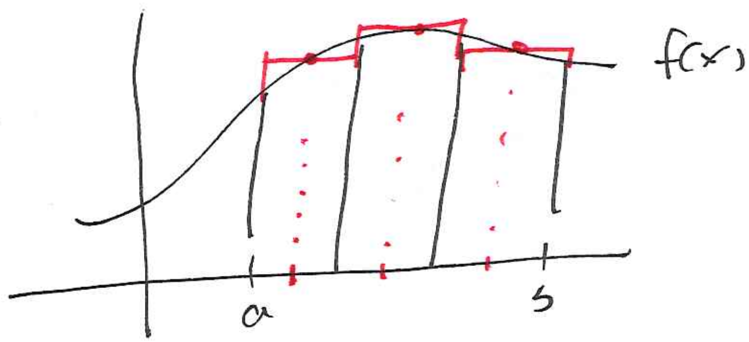
Trapezoid: $\int_a^b f(x)\,dx \approx \Delta x\left(\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right)$

where $\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$

Midpoint: $\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\Delta x$

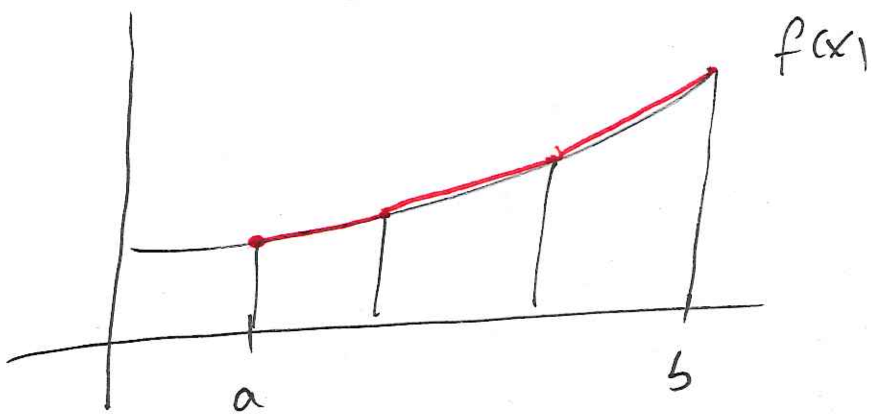
2 March 2017

Midpoint Rule (p. 559)



Approximating f(x)
by a constant
(in each interval)

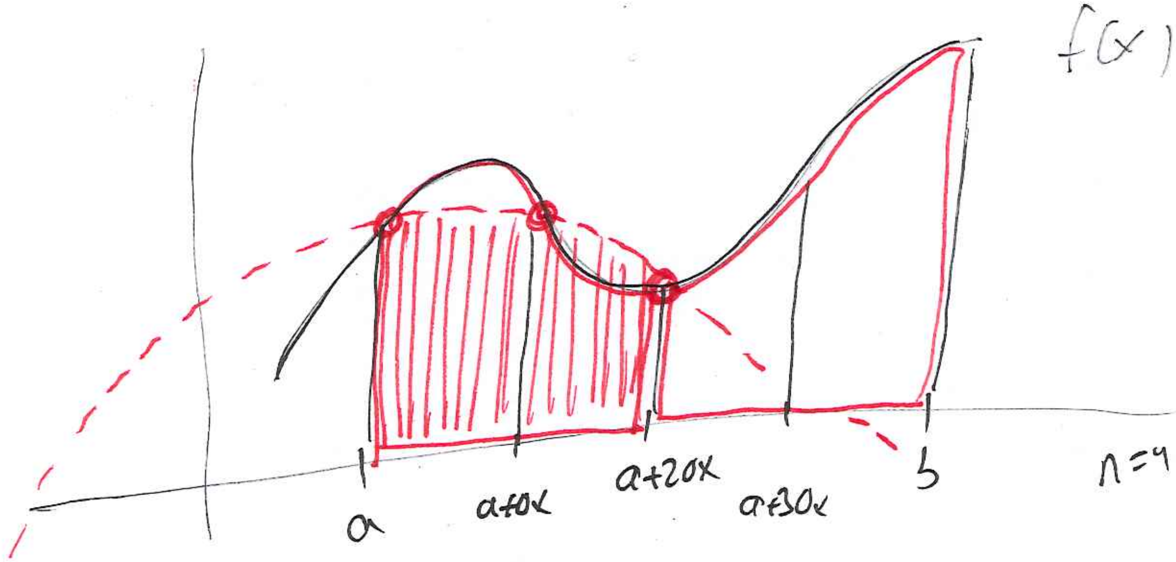
Trapezoid Rule (p. 560)



Approximating f(x)
by a line
(in each interval)

Simpson's Rule:

Approx f(x) by a parabola (in each interval)



Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 4f(x_{n-1}) + f(x_n) \Big]$$

$\Delta x = \frac{b-a}{n}$

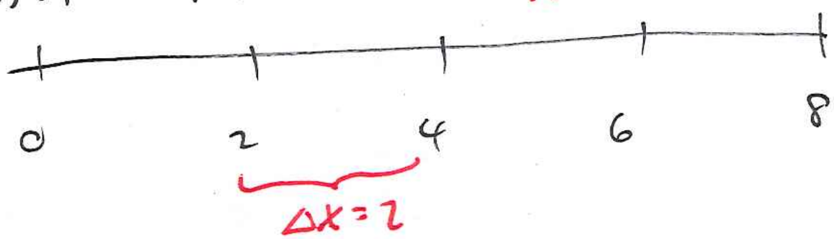
$\boxed{\text{only when } n \text{ even}}$

(ex) Use Simpson's Rule, $n=4$ intervals

$$\int_0^8 \frac{1}{1+x^2}\,dx \approx \frac{2}{3}\left[1+4\cdot\frac{1}{5}+2\cdot\frac{1}{17}+4\cdot\frac{1}{37}+\frac{1}{65}\right]$$

$f(0)=1$ $\quad f(2)=\frac{1}{5}$ $\quad f(4)=\frac{1}{\cancel{16}\,17}$ $\quad f(6)=\frac{1}{37}$ $\quad f(8)=\frac{1}{65}$

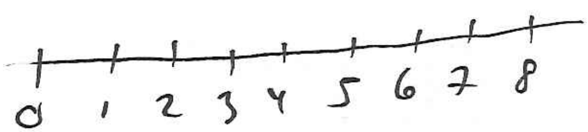
$$\Delta x=\frac{b-a}{n}=\frac{8}{4}=2$$



(ex) Use Simpson's Rule, 8 intervals

$$\int_0^8 e^{x^2}\,dx \approx \frac{1}{3}\left[e^0+4e^1+2e^4+4\cdot e^9+2e^{16}+4e^{25}+2e^{36}+4e^{49}+e^{64}\right]$$

$$\Delta x=\frac{b-a}{n}=\frac{8-0}{8}=1$$



Formulas for Error: formula sheet

Theorem 7.2, p565

(ex) Find error involved with approximating $\int_0^1 \sin(2x)\,dx$ using 10 intervals (all 3 methods)

$b = 1$ $\quad (b-a) = 1$

$a = 0$ $\quad \Delta x = \frac{b-a}{n} = \frac{1}{10}$

$n = 10$

$f(x) = \sin(2x)$

$f'(x) = 2\cos(2x)$

$\boxed{f''(x) = -4\sin(2x)}$ $\quad |-4\sin(2x)| \le 4$

$f'''(x) = -8\cos(2x)$

$\boxed{f^{(4)}(x) = 16\sin 2x}$ $\quad |16\sin(2x)| \le 16$

$4 = k$

$16 = K$

Using Midpoint: $|\text{Error}| \le \frac{k(b-a)}{24}(\Delta x)^2 = \frac{4\cdot 1}{24}\left(\frac{1}{10}\right)^2 = \frac{1}{6}\cdot\frac{1}{100} = \boxed{\frac{1}{600}}$

Using Trapezoid: $|\text{Error}| \le \frac{k(b-a)}{12}(\Delta x)^2 = \frac{4\cdot 1}{12}\left(\frac{1}{10}\right)^2 = \frac{1}{3}\cdot\frac{1}{100} = \boxed{\frac{1}{300}}$

Possible: (eg) error using MP: $\frac{1}{600}$

error using ⏢ : $\frac{1}{1000}$

Error
Using Simpson's Rule: $\leq \dfrac{K(b-a)}{180}(\Delta x)^4$

$$= \frac{16(1)}{180}\left(\frac{1}{10}\right)^4 = \frac{1}{112{,}500}$$

(ex) What is the error involved with Simpson's Rule using 6 intervals? approximating $\int_1^2 \frac{1}{x}\,dx$

$$E_S \leq \frac{K(b-a)}{180}(\Delta x)^4 = \frac{24(2-1)}{180}\Big(\underbrace{\frac{2-1}{6}}_{\Delta x}\Big)^4$$

K : upper-bound on $|f^{(4)}(x)|$

$$= \frac{24}{180 \cdot 6^4} \stackrel{?}{=} \frac{1}{38{,}880}$$

$f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -x^{-2}$

$f''(x) = 2x^{-3}$

$f'''(x) = -6x^{-4}$

$f^{(4)}(x) = 24x^{-5} = \frac{24}{x^5}$

$\left|\frac{24}{x^5}\right| \leq \frac{24}{1} = 24$

Use $K = 24$

Note: $\int_1^2 \frac{1}{x}\,dx = \ln 2 - \ln 1 = \boxed{\ln 2}$ exact

Suppose you want to approx $\ln 2 = \int_1^2 \frac{1}{x}\,dx$ using midpoint rule, your error should be at most 10^{-4}.

How many intervals do you need?

$$E_M \leq \frac{k(b-a)}{24}(\Delta x)^2 \underset{\text{WANT}}{\leq} 10^{-4}$$

$f(x) = \frac{1}{x}$

$f'(x) = \frac{-1}{x^2}$

$\boxed{f''(x) = \frac{2}{x^3}}$ → when x is between 1 & 2,

$|f''(x)| = \left|\frac{2}{x^3}\right| \leq \frac{2}{1} = 2$

Use: $\boxed{k=2}$

$$\frac{k(b-a)}{24}(\Delta x)^2 \overset{\text{want}}{\leq} 10^{-4}$$

$$\frac{2(\cancel{2-1})}{24}\left(\frac{\cancel{b-a}^{\,1}}{n}\right)^2 \leq \frac{1}{10^4}$$

$$\frac{1}{12}\cdot\frac{1}{n^2} \leq \frac{1}{10^4}$$

$$12n^2 \geq 10^4$$

$$n^2 \geq \frac{10^4}{12}$$

$$n \geq \sqrt{\frac{10^4}{12}} = \frac{100}{\sqrt{12}} \approx 28.8$$

use $\boxed{29}$ intervals.

→ Suggested Probs §7.7

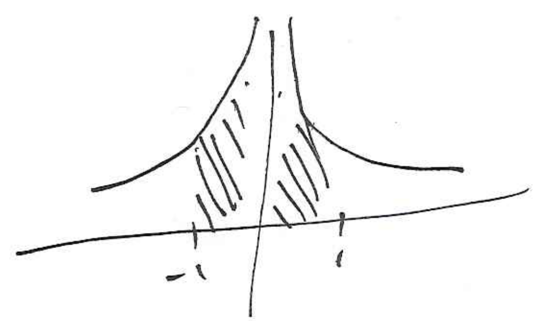
§7.8 Improper Integrals

Two ways for an integral to be improper:

- infinite interval of integration
- integrand ($f(x)$) not bounded on region of integration

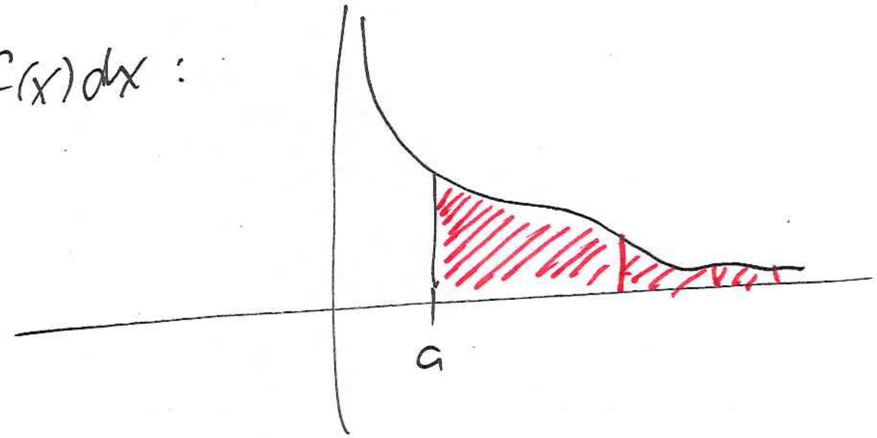
example: $\int_1^\infty \frac{1}{x^2}\,dx$ improper

$\int_0^1 \frac{1}{x^2}\,dx$, $\int_{-1}^1 \frac{1}{x^2}\,dx$ improper



Infinite Interval

We use a limit $\int_a^\infty f(x)\,dx$:



ex $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$

$$= \lim_{b\to\infty}\left[\frac{-1}{x}\Big|_1^b \right]$$

$$= \lim_{b\to\infty}\left[\frac{-1}{b} - \frac{-1}{1}\right] = \lim_{b\to\infty}\left[\frac{-1}{b}+1\right] = \boxed{1}$$

ex $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx \neq \lim_{a\to\infty}\int_{-a}^{a}\frac{1}{1+x^2}\,dx$

$$\|$$

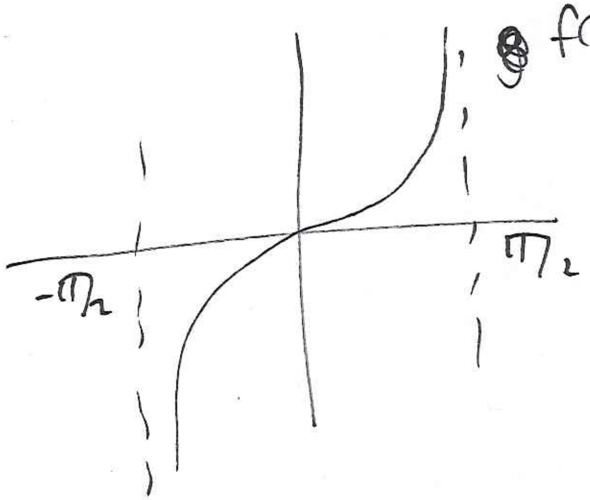
$$\int_{-\infty}^{0}\frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty}\left[\int_a^0 \frac{1}{1+x^2}\,dx\right] + \lim_{b\to\infty}\left(\int_0^b \frac{1}{1+x^2}\,dx\right)$$

$$= \lim_{a\to-\infty}\left[\overset{0}{\cancel{\arctan 0}} - \arctan a\right] + \lim_{b\to\infty}\left[\arctan b - \overset{0}{\cancel{\arctan 0}}\right]$$

$$= -\left[-\pi/2\right] + \pi/2 = \pi/2 + \pi/2 = \boxed{\pi}$$

Aside: $\arctan(x) = y$

means: $\tan(y) = x$



As $y \to \pi/2$, $\tan(y) \to \infty$ $\Big\}$ "$\arctan \infty$" "=" "$\pi/2$"

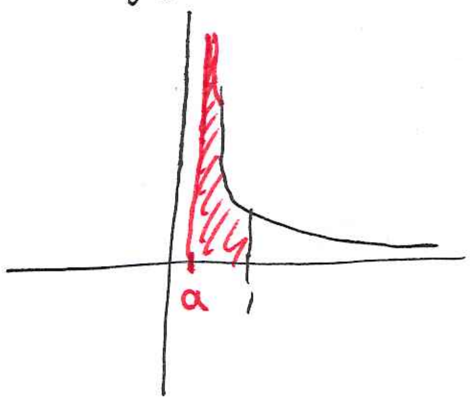
$$\lim_{x \to \infty} \arctan x = \pi/2$$

$$\lim_{x \to \pi/2^-} \tan x = \infty$$

$$\lim_{x \to -\infty} \arctan x = -\pi/2$$

Unbounded Function

ex) $$\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\left[\int_a^1 \frac{1}{x^2}\,dx\right] = \lim_{a\to 0^+}\left[\frac{-1}{x}\Big|_a^1\right]$$



$$= \lim_{a\to 0^+}\left[\frac{-1}{1} - \frac{-1}{a}\right]$$

$$= \lim_{a\to 0^+}\left[-1 + \frac{1}{a}\right] = \infty$$

We say this integral diverges
(limit doesn't exist!)

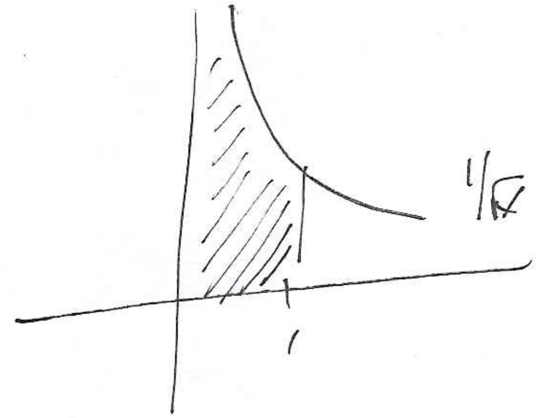
By the way: If limit gives a finite number, we say the integral converges.

(ex) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\left[\int_a^1 x^{-1/2}\,dx\right]$

$\displaystyle = \lim_{a\to 0^+}\left[\, 2x^{1/2}\Big|_a^1 \right]$

$\displaystyle = \lim_{a\to 0^+}\left[2\sqrt{1} - 2\sqrt{a}\right]$

$= \boxed{2}$



Improper Integrals

ex $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+}\left[\int_a^1 \frac{1}{x}\,dx\right] = \lim_{a\to 0^+}\left[\cancel{\ln 1}^{\,0} - \ln a\right] = \infty$

DIVERGES

$\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\int_1^b \frac{1}{x}\,dx\right] = \lim_{b\to\infty}\left[\ln b - \cancel{\ln 1}^{\,0}\right] = \infty$

DIVERGES

ex $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[\int_1^b x^{-2}\,dx\right] = \lim_{b\to\infty}\left[-x^{-1}\Big|_1^b\right]$

CONVERGES $\displaystyle = \lim_{b\to\infty}\left[-\frac{1}{b} - \frac{-1}{1}\right] = \lim_{b\to\infty}\left[1 - \cancel{\frac{1}{b}}^{\,0}\right] = 1$

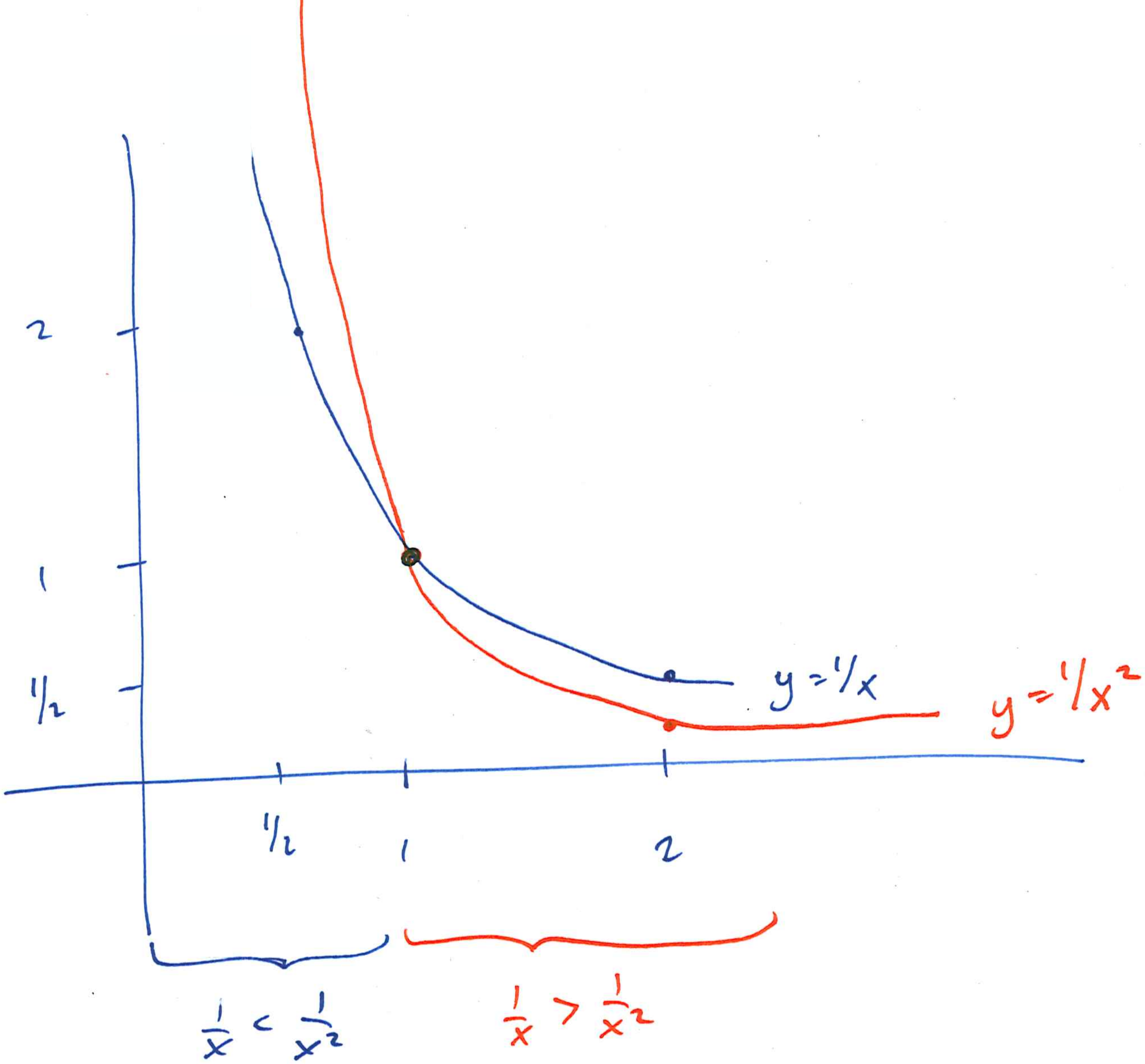
$\displaystyle\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to 0^+}\left[\frac{-1}{x}\Big|_a^1\right] = \lim_{a\to 0^+}\left[\frac{-1}{1} - \underbrace{\frac{-1}{a}}_{\to\infty}\right] = \infty$

DIVERGES

$f(x) = \frac{1}{x^p}$, $p > 0$ constant

$\frac{1}{(1/2)^2} = \frac{1}{1/4} = 4$

$\frac{1}{2^2} = \frac{1}{4}$



ex $\int_0^1 \frac{1}{x^{0.999}}\,dx = \lim_{a\to 0^+}\left[\int_a^1 x^{-0.999}\,dx\right] =$

$\lim_{a\to 0^+}\left[1000\,x^{0.001}\Big|_a^1\right] = \lim_{a\to 0^+}\left[1000 - 1000\cdot \cancel{a^{0.001}}_{\to 0}\right] = 1000$

So: $\int_0^1 \frac{1}{x^{0.999}}\,dx$ converges

ex $\int_0^1 \frac{1}{x^{1.001}}\,dx = \lim_{a\to 0^+}\left[\int_a^1 x^{-1.001}\,dx\right] = \lim_{a\to 0^+}\left[-1000x^{-0.001}\Big|_a^1\right]$

$= \lim_{a\to 0^+}\left[-1000 + 1000\,a^{-0.001}\right]$

$= \lim_{a\to 0^+}\left[-1000 + \frac{1000}{a^{0.001}}\right]_{\to 0} = \infty$

$0.001 = \frac{1}{1000}$

$\frac{1}{0.001} = \frac{1}{1/1000} = 1000$

So: $\int_0^1 \frac{1}{x^{1.001}}\,dx$ DIVERGES p-test:

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{converges} & \text{if } p < 1 \\ \text{diverges} & \text{if } p \geq 1 \end{cases}$$

p positive constant

$$\int_1^\infty \frac{1}{x^p}\,dx : \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$$

ex $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$: DIVERGES (by p-test)

$p = 1/2 < 1$

ex $\int_0^1 \frac{1}{x^{2.7}}\,dx$ DIVERGES

$p = 2.7 > 1$

(ex) $\displaystyle\int_0^{17} \frac{1}{\sqrt{x}}\,dx$ Conv or Div?

$$= \underbrace{\int_0^1 \frac{1}{\sqrt{x}}\,dx}_{\text{determines conv/div}} + \underbrace{\int_1^{17} \frac{1}{\sqrt{x}}\,dx}_{\text{some number (not improper)}}$$

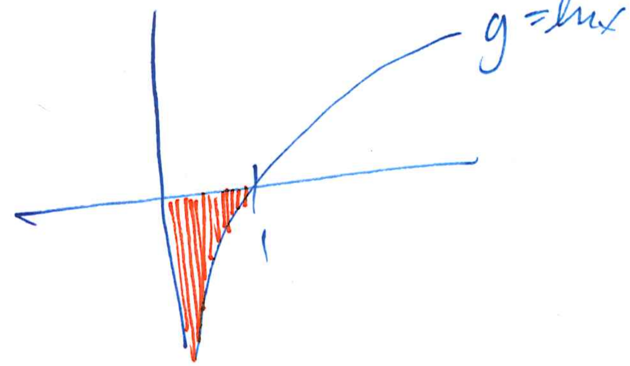
$p = 1/2 < 1$

converge

So: $\displaystyle\int_0^{17} \frac{1}{\sqrt{x}}\,dx$ converges.

LOL nevermind

ex $\displaystyle\int_0^1 \ln x \, dx$



Evaluate.

$$\int 1 \cdot \ln x \, dx = x \ln x - \int x \cdot \tfrac{1}{x} \, dx = x \ln x - \int 1 \, dx$$

$$= \boxed{x \ln x - x} + C$$

$u : \ln x \qquad du : \tfrac{1}{x}\, dx$

$dv : 1\, dx \qquad v : x$

$$\int_0^1 \ln x \, dx = \lim_{a \to 0^+} \left[\int_a^1 \ln x \, dx \right] = \lim_{a \to 0^+} \left[(1 \overset{0}{\cancel{\ln 1}} - 1) - (a \ln a - a) \right]$$

$$= \lim_{a \to 0^+} \left[-1 - a \ln a + a \right] = \lim_{a \to 0^+} \left[-1 - \underset{0}{a} (\underset{-\infty}{\ln a - 1}) \right]$$

indeterminate form

$$= \lim_{a \to 0^+} \left[-1 - \underbrace{\frac{\ln a - 1}{1/a}}_{\frac{\infty}{\infty}} \right] = \lim_{a \to 0^+} \left[-1 + \frac{1/a}{+1/a^2} \right] = \lim_{a \to 0^+} \left[-1 + a \right] = \boxed{-1}$$

use L'Hospital

Ch 7.9 Differential Equations

(ex) $\boxed{y' = e^x}$ and $y(0) = 2$. What is y?

1st order differential equation

$y = \int e^x dx = e^x + C$, $\quad y = e^x + C$

$x=0$: $2 = e^0 + C$

$2 = 1 + C$

$C = 1$

So: $\boxed{y = e^x + 1}$

2nd order differential equation

(ex) $\boxed{y''(t) = 12t + 1}$, $y(0) = 1$, $y(1) = 10$. Find y

$y'(t) = \int 12t+1\, dt = 6t^2 + t + C$

$y(t) = \int (6t^2 + t + C)\, dt = 2t^3 + \frac{1}{2}t^2 + C\cancel{t} + D$

$\boxed{y = 2t^3 + \frac{1}{2}t^2 + 6.5t + 1}$

$y(t) = 2t^3 + \frac{1}{2}t^2 + Ct + D$

$1 = D \quad (y(0) = 1)$

$10 = 2 + \frac{1}{2} + C + D$

$10 = 2 + \frac{1}{2} + C + 1$

$C = 6.5$

(ex) $\boxed{y' = ky + b}$

where k, b constants
y function of ~~x~~ t

General Solution:

$$y = Ce^{kt} - b/k$$

for some C - constant

(ex) $y' = 3y + 7$, $y(2) = 5$
what is y?

$$y = Ce^{3t} - 7/3 \quad \text{for some } C$$

[$y(2)=5$] $5 = C \cdot e^{3 \cdot 2} - 7/3$ find C

$$\frac{22}{3} = C \cdot e^{6}$$

$$\frac{22}{3 \cdot e^{6}} = C$$

So: $y = \frac{22}{3e^{6}} \cdot e^{3t} - 7/3$

$$\boxed{y = \frac{22}{3} e^{3t-6} - 7/3}$$

Check: $y' = 3y + 7$

$$y' = \frac{22}{3} \cdot e^{3t-6} \cdot 3 = 22e^{3t-6}$$

$$3y + 7 = 3(\underbrace{\frac{22}{3} e^{3t-6} - 7/3}_{y}) + 7$$

$$= 22e^{3t-6} - 7 + 7 = 22e^{3t-6}$$

TRUE: $y' = 3y + 7$ for this y

Check: $y(2) = 5$

$$y(t) = \frac{22}{3}e^{3t-6} - 7/3$$

$$y(2) = \frac{22}{3}e^{0} - 7/3 = \frac{22}{3} - \frac{7}{3} = \frac{15}{3} = 5$$ ✓

Differential Equations

Which of the following satisfies

↳ makes true

$$\frac{dy}{dx} + x^2 - 1 = y$$

Ⓐ $y = x^2 + 1$ $\quad (2x) + x^2 - 1 = x^2 + 1$

$x^2 + 2x - 1 = x^2 + 1$ FALSE

not a solution

Ⓑ $y = x^2 + 2x + 1$ $\quad (2x+2) + x^2 - 1 = x^2 + 2x + 1$

$x^2 + 2x + 1 = x^2 + 2x + 1$ TRUE

a solution to our diff. eq.

Ⓒ $y = -\frac{1}{3}x^3 + x$ $\quad (-\cancel{x^2} + 1) + \cancel{x^2} - 1 = -\frac{1}{3}x^3 + x$

$0 = -\frac{1}{3}x^3 + x$ FALSE

not a solution

ex $\quad \dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = x+2$

Which is a solution?

~~(A)~~ $y = e^x$ $\quad e^x + 2e^x + e^x = x+2$

$4e^x = x+2$ $\quad$ FALSE

~~(B)~~ $y = x^2$ $\quad 2 + 2(2x) + x^2 = x+2$ $\quad$ FALSE

$y' = 2x$ $\quad x^2 + 4x + 2 = x+2$

$y'' = 2$

✓ (C) $y = x$ $\quad 0 + 2(1) + x = x+2$ $\quad$ TRUE

$y' = 1$ $\quad 2 + x = x+2$

$y'' = 0$

~~(D)~~ $y = x+1$

$y' = 1$ $\quad 0 + 2\cdot(1) + (x+1) = x+2$ $\quad$ FALSE

$y'' = 0$ $\quad x+3 = x+2$

Separable Differential Equations

ex $\dfrac{dy}{dx} = y^2 x, \qquad y(0) = 1$

long

$$\frac{1}{y^2} \cdot \frac{dy}{dx} = x$$

$$\int \frac{1}{y^2} \frac{dy}{dx}\, dx = \int x\, dx$$

$$\frac{dy}{dx} = \frac{dy}{dx}$$

$$dy = \left(\frac{dy}{dx}\right) dx$$

$$\int \frac{1}{y^2} dy = \int x\, dx$$

$$\int \frac{1}{y^2} y'\, dx \rightarrow dy$$

$$y' = \frac{dy}{dx}$$

$$y'\,dx = dy$$

short

$$\frac{1}{y^2} \cdot \frac{dy}{dx} = x$$

$$\frac{1}{y^2}\, dy = x\, dx$$

$$\int \frac{1}{y^2} dy = \int x\, dx$$

$$\frac{-1}{y} = \frac{1}{2}x^2 + C$$

find C

If $x=0$, $y=1$

$$\frac{-1}{1} = 0 + C \rightarrow \boxed{C = -1}$$

$$\frac{-1}{y} = \frac{1}{2}x^2 - 1$$

find y

$$\frac{1}{y} = -\frac{1}{2}x^2 + 1$$

$$\boxed{y = \frac{1}{-\frac{1}{2}x^2 + 1}}$$

ex $\dfrac{dy}{dx} = e^{x-y}$

$dy = e^{x-y}\,dx$

$dy = \dfrac{e^x}{e^y}\,dx$

$e^y\,dy = e^x\,dx$

$\int e^y\,dy = \int e^x\,dx$

$e^y = e^x + C$

$\boxed{y = \ln(e^x + C)}$

ex $\dfrac{dy}{dx} = y(4x^3 - 1)$, $\boxed{y(0) = -2}$

$\frac{1}{y}\,dy = (4x^3 - 1)\,dx$

$\int \frac{1}{y}\,dy = \int (4x^3 - 1)\,dx$

$\ln|y| = x^4 - x + C$

Find C:

If $x = 0$, $y = -2$

$\ln|-2| = 0^4 - 0 + C$

$\ln 2 = C$

$\ln|y| = x^4 - x + \ln 2$

$|y| = e^{x^4 - x + \ln 2}$

$\boxed{y = -e^{x^4 - x + \ln 2}}$

Find y

If $y > 0$, $|y| = y$

If $y < 0$, $|y| = -y$

ex $\frac{dy}{dx} \cdot \sqrt{y(9-x^2)} = -2x$ where $y > 0$ for all x

$$\frac{dy}{dx} \cdot \sqrt{y} \cdot \sqrt{9-x^2} = -2x$$

$$\int \sqrt{y}\, dy = \int \frac{-2x}{\sqrt{9-x^2}}\, dx$$

$$\int y^{1/2}\, dy = \int \frac{\overbrace{-2x\, dx}^{du}}{\sqrt{9-x^2}}$$

$u = 9-x^2$
$du = -2x\,dx$

$$\frac{2}{3} y^{3/2} = \int \frac{1}{\sqrt{u}}\, du$$

$$\frac{2}{3} y^{3/2} = \int u^{-1/2}\, du$$

$$\frac{2}{3} y^{3/2} = 2u^{1/2} + C$$

$$\frac{2}{3} y^{3/2} = 2\sqrt{9-x^2} + C$$

$$y^{3/2} = 3\sqrt{9-x^2} + C$$

$$\boxed{y = \left[3\sqrt{9-x^2} + C\right]^{2/3}}$$

Note:

$$y = \sqrt[3]{3\sqrt{9-x^2} + C}^{\,2}$$

ex $\sec x \frac{dy}{dx} = y^3$

$$\frac{1}{\cos x} \cdot \frac{dy}{dx} = y^3$$

$$\int y^{-3}\,dy = \int \cos x \, dx$$

$$-\frac{1}{2} y^{-2} = \sin x + C$$

$$\frac{-1}{2y^2} = \frac{\sin x + C}{1}$$

$$\frac{2y^2}{-1} = \frac{1}{\sin x + C}$$

$$2y^2 = \frac{-1}{\sin x + C}$$

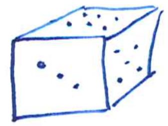
$$y^2 = \frac{-1}{2\sin x + C}$$

① $y = \sqrt{\frac{-1}{2\sin x + C}}$

$\boxed{\text{OR}}$

② $y = -\sqrt{\frac{-1}{2\sin x + C}}$

Probability



- Probability: Number from 0 to 1
  Interpret: liklihood an event will happen

0: no matter how many tries, never happens

1: (100%) no many tries, always happens

1/3: if try lots & lots of times, event happens in ~1/3

$\left(\lim_{\#\text{tries} \to \infty} \left[\frac{\#\text{ tries where event happened}}{\#\text{ tries total}} \right] \right)$

X: dice roll

Notation: (conventional)

Event: capital letter, X
(trial)

x: 4

Value event might take: lower-case letter, x

* $\boxed{Pr(X=x)}$: Probability that trial X gives a value of x

X : rolling a dice

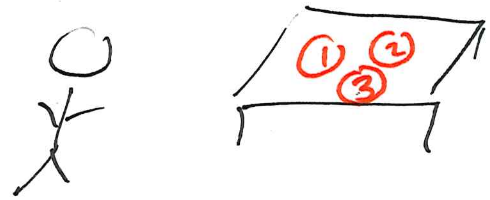
$$\underbrace{Pr(X=6)}_{\text{Prob. that I roll a 6}}$$

$$Pr(X=1)$$

---

ex

X : selection of participant



① : your product

② : competitors'

③ : products

What is $\underbrace{Pr(X=1)}_{\text{probability that selection of participant is ①}}$

$$\Pr(X = x \text{ or } X \neq x) = 1$$

$$\Pr(X = x) = 1 - \Pr(X \neq x)$$

(ex) If an unfair coin flips Heads 70% of the time

$\Pr(X = H) = 0.7$

Then: $\Pr(X = T) = 0.3$

discrete: "listable"
possible outcomes of an event

• Roll 3 dice, add values	Outcomes: 3, 4, 5, ..., 18	DISCRETE
• Choose a whole # from 1 to 10	Outcomes: 1, 2, 3, ..., 10	DISCRETE
• Choose any real # from 1 to 10	Outcomes: [1, 10] exist along a continuum	NOT DISCRETE CONTINUOUS
• The exact age of a person at noon today	Outcom: [0, 200]	NOT DISCRETE
• Amount of oil spilled in an oil spill	Ambiguous # molecules whole # (discrete)	weight could be any #, [0, N] N: weight of earth (not discrete)